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RESEARCH DEPARTMENT REPORT

HADAMARD TRANSFORMATION: Walsh spectral analysis of television signals

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Research Department, Engineering Division
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HADAMARD TRANSFORMATION: WALSH SPECTRAL ANALYSIS OF TELEVISION SIGNALS

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Summary

The characteristics of Walsh-Hadamard transformation systems are examined with a view to reducing the data rate of digital television signals. Transformed signals have been measured using several different transformation arrangements by means of a Walsh spectrum analyser, in which television signals are correlated with Walsh functions.

It is found that a slightly higher data rate is required to describe the transform coefficients with linear coding than is needed for the original signal. However, a more efficient encoding of the transform could be provided using non-linear quantisation or adaptive bit-rate reduction systems. Two-dimensional transformation and the use of sampling frequencies which concentrate the energy of colour signals in the transform could improve the performance of these bit-rate reduction systems.

Issued under the authority of

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HADAMARD TRANSFORMATION: WALSH SPECTRAL ANALYSIS OF TELEVISION SIGNALS

C.K.P. Clarke, B.Sc.(Eng.), A.C.G.I.

1. Introduction

The conversion of television signals into pulse code modulation (p.c.m.) greatly improves their immunity to distortion. However, the p.c.m. signal typically consists of samples taken at a rate of 13·3 MHz, each of which is coded using an eight-digit binary word.[1] Therefore, the data rate of the p.c.m. signal is 106·4 Mbit.s^{-1}. A reduction of this data rate without noticeable degradation of picture quality would produce a corresponding saving in the capacity required for distribution links and the amount of magnetic tape used in recording.

All methods of bit-rate reduction convert the p.c.m. signal into a new set of codewords containing, on average, fewer binary digits. After transmission, an inverse conversion is used to return the signals to p.c.m., although, because of the bit-rate reduction, some combinations of p.c.m. words cannot be reproduced without error. If picture quality is to be preserved, then either the effect of the errors must not be noticeable, or the combinations of p.c.m. words which would produce noticeable errors must never occur.

For television signals, more efficient coding methods can be found by first identifying the respects in which the p.c.m. signals are redundant and then designing codes which reduce the redundancy. The following two examples illustrate redundancy in the p.c.m. signal.

First, eight-bit p.c.m. words are required to avoid 'contouring' in plain areas of the picture, but fewer bits could represent detailed areas adequately. This is because the detail disguises quantising errors. Secondly, in plain areas adjacent samples are similar in value, so that only those parts of the code that represent small differences change from word to word.

This suggests a code which transmits small differences, which occur in plain areas of picture, with high precision and large differences, which occur in detailed areas, with reduced precision. Differential pulse-code modulation systems use this principle.[2]

Another approach is to separate the parts of the signal representing the plain and detailed areas so that each may be described with the optimum number of bits. Plain areas correspond to low frequencies in the signal and detailed areas correspond to high frequencies. Therefore, the two types of signal could be separated by Fourier transformation. However, Hadamard transformation also separates signals representing plain and detailed areas of picture and involves simpler calculations.

Both Fourier and Hadamard transformations represent the signal as a set of amplitudes of orthogonal functions. In Fourier transformation, these are harmonically related sinusoidal functions, whereas in Hadamard transformation they are rectangular waveforms. In the most commonly used Hadamard transformation, known as Walsh-Hadamard transformation, the orthogonal functions are sets of Walsh functions.

Walsh functions may take only the values +1 and −1, and it is this property which makes Hadamard transformation easier to implement than Fourier transformation. Walsh functions are distinguished by the number of transitions between these values in a given interval; this number is known as the sequency of the function. When placed in order of increasing sequency, the set of transform coefficients of the Walsh-Hadamard transformation is known as the sequency spectrum of the signal.

A large number of different sequency spectra can be produced by arranging the television signals for transformation in different ways. The object of the work described in this report was to find the transformation arrangements which are most beneficial for subsequent bit-rate reduction. For some simple methods, the savings to be made can be calculated by considering the properties of the transformation and making measurements on the sequency spectrum of the signals. This report first presents some of the theory of Hadamard transformation as a means to the bit-rate reduction of transformed signals. This is followed by a description of the Walsh correlation methods used to measure the sequency spectra of different transformations. Finally, the results for different transformation arrangements are presented and considered.

2. Hadamard transformation

2.1. Theory of the transformation

Hadamard transformation is defined by the matrix equation:

$$\begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \dots h_{1n} \\ h_{21} & h_{22} & \dots h_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ h_{n1} & h_{n2} & \dots h_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad (1)$$

or $\mathbf{c} = \mathbf{Hx}$

where $\mathbf{x}$ and $\mathbf{c}$ represent input and output n-dimensional vectors and $\mathbf{H}$ represents an $(n \times n)$ Hadamard matrix. A Hadamard matrix has two properties: it is orthogonal, that is:

$$\mathbf{HH}^T = n\mathbf{I} \qquad (2)$$

where $\mathbf{H}^T$ represents the transpose of $\mathbf{H}$, and $\mathbf{I}$ represents the $(n \times n)$ unit matrix; the elements of the Hadamard matrix may take only the values +1 or −1, that is:

$$h_{ij} = \pm 1 \qquad i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n.$$

Some Hadamard matrices are symmetric, that is:

$$\mathbf{H} = \mathbf{H}^T$$

so that Equation (2) becomes:

$$\mathbf{HH} = n\mathbf{I} \tag{3}$$

Then in the inverse transformation

$$\mathbf{x} = \mathbf{H}^{-1}\mathbf{c}$$

the inverse Hadamard matrix $\mathbf{H}^{-1}$ is given by:

$$\mathbf{HHH}^{-1} = n\mathbf{H}^{-1} \qquad \text{Using (3)}$$

$$\therefore \qquad \mathbf{H}^{-1} = \frac{1}{n}\mathbf{H}$$

$$\text{and} \qquad \mathbf{x} = \frac{1}{n}\mathbf{Hc} \tag{4}$$

So, for symmetric Hadamard matrices, the inverse transformation is the same as the forward transformation except that it includes division by the constant n.

A particular set of symmetric Hadamard matrices, known as Walsh-Hadamard matrices, is formed by the direct products of the matrix $\mathbf{H}_2$ with itself, that is:

$$\mathbf{H}_{2^p} = \mathbf{H}_2^{(1)} \otimes \mathbf{H}_2^{(2)} \otimes \ldots \otimes \mathbf{H}_2^{(p)}$$

$$\text{where} \quad \mathbf{H}_2^{(r)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad r = 1, 2, \ldots, p$$

and $\mathbf{H}_{2^p}$ is the Walsh-Hadamard matrix of order $(2^p \times 2^p)$. The direct or Kronecker product operation,[3] denoted $\otimes$, is defined for a $(u \times u)$ matrix $\mathbf{A}$ and a $(v \times v)$ matrix $\mathbf{B}$ as:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \ldots a_{1u}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \ldots a_{2u}\mathbf{B} \\ \cdot & \cdot \qquad \cdot \\ \cdot & \cdot \qquad \cdot \\ \cdot & \cdot \qquad \cdot \\ a_{u1}\mathbf{B} & a_{u2}\mathbf{B} \ldots a_{uu}\mathbf{B} \end{bmatrix}$$

where $\mathbf{A} \otimes \mathbf{B}$ is a $(uv \times uv)$ matrix. So $(n \times n)$ Walsh-Hadamard matrices can be constructed where $n = 2^p$ and p is a positive integer. The (8 x 8) Walsh-Hadamard matrix ($p = 3$) is shown in Table 1.

TABLE 1

Walsh-Hadamard Matrix of Order (8 x 8)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Equation (1) may be re-written as:

$$c_i = \sum_{j=1}^{n} h_{ij}x_j \qquad i = 1, 2, \ldots, n \tag{5}$$

which shows c_i as a measure of the correlation between the input vector $\mathbf{x}$ and a discrete, two-valued function defined by the ith row of a Hadamard matrix. The functions formed by the rows of the Walsh-Hadamard matrix of order $(n \times n)$ are the set of Walsh functions of period n. Fig. 1 shows the set of Walsh functions of period 8 arranged in an order corresponding to the rows of the Walsh-Hadamard matrix. So, the vector of coefficients produced

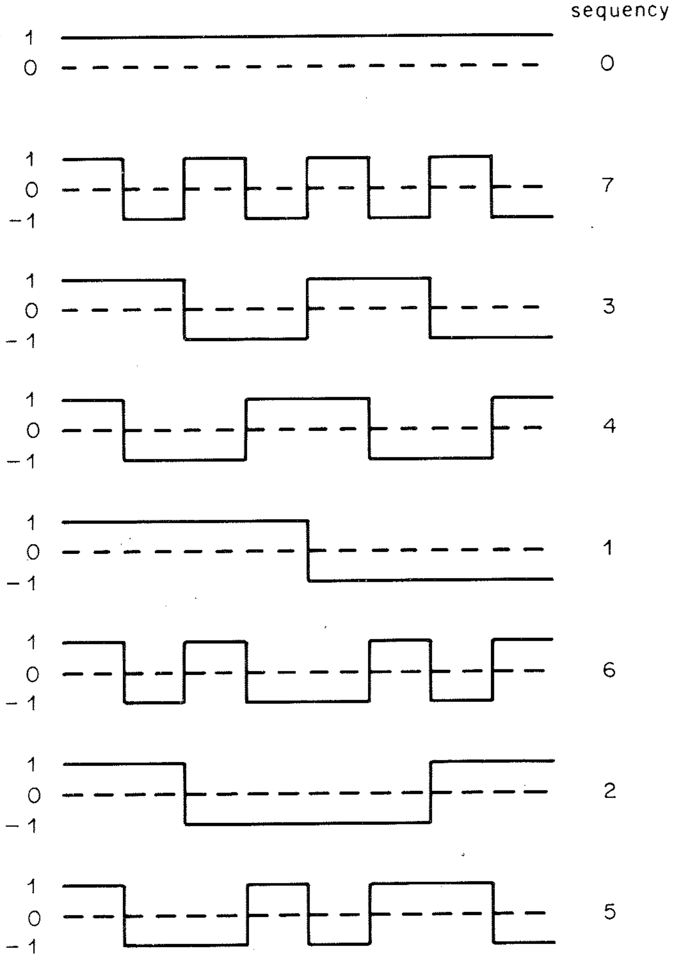

Fig. 1 - The set of Walsh functions of period 8

by Walsh-Hadamard transformation describes the input data in terms of the amplitudes of a set of Walsh functions.

Fourier analysis of Walsh functions reveals a linear relationship between the sequency of each Walsh function (the number of transitions between +1 and −1 in a period) and the frequency of the largest sinusoidal component of the Walsh function. For this reason, in the Walsh-Hadamard transform or sequency spectrum of a signal, the low-sequency components tend to represent low frequencies in the signal and the high-sequency components tend to represent high frequencies in the signal.

Other Hadamard matrices do not have this property. For example, the matrix shown in Table 2 is formed from cyclically shifted versions of a maximal-length pseudo-random sequence to which an extra row and an extra column of 1's have been added.

TABLE 2

Pseudo-Random Hadamard Transform Matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \end{bmatrix}$$

Transformation with this matrix describes input data in terms of the amplitudes of the set of functions shown in Fig. 2. Individually, these functions show no regular structure, so it is difficult to interpret the transform coefficients as a spectrum.

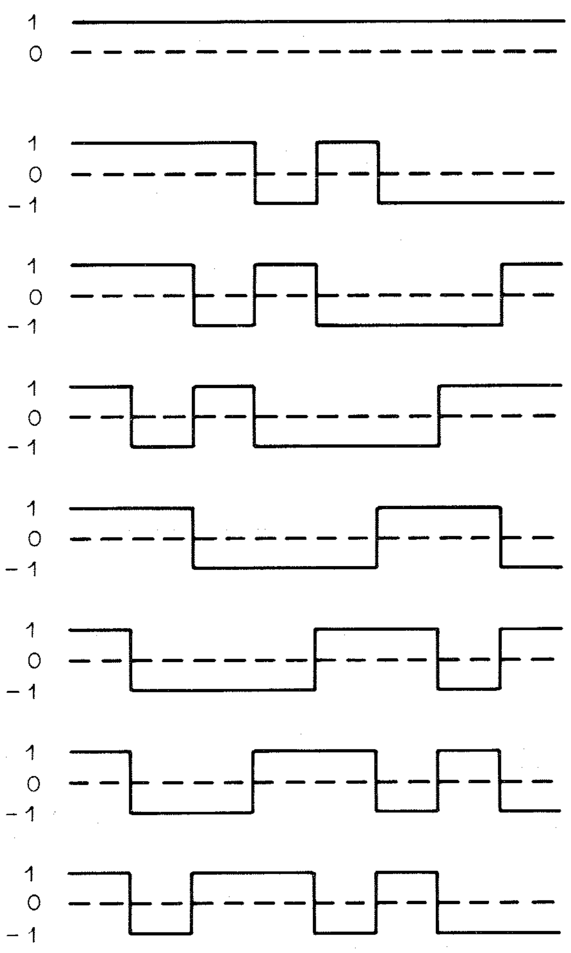

Fig. 2 - A set of orthogonal functions of period 8 formed from a maximal-length pseudo-random sequence

2.2. Transformation example

Walsh-Hadamard transformation and inverse transformation will now be demonstrated using the portion of a television signal shown in Fig. 3.

The digital signals to be transformed consist of a series of eight-digit binary numbers. These can represent 256 discrete voltage levels in the corresponding analogue waveform which will be labelled 0 to 255 for convenience. The amplitudes of the signal of Fig. 3 at the marked sampling points are shown in Table 3.

TABLE 3

Values of the Samples Shown in Fig. 3

Sample	1	2	3	4	5	6	7	8
Amplitude	105	74	75	107	137	158	167	165

These numbers represent a sampled-and-held and quantised version of the analogue signal, as shown in Fig. 4.

For transformation, a set of n samples must be

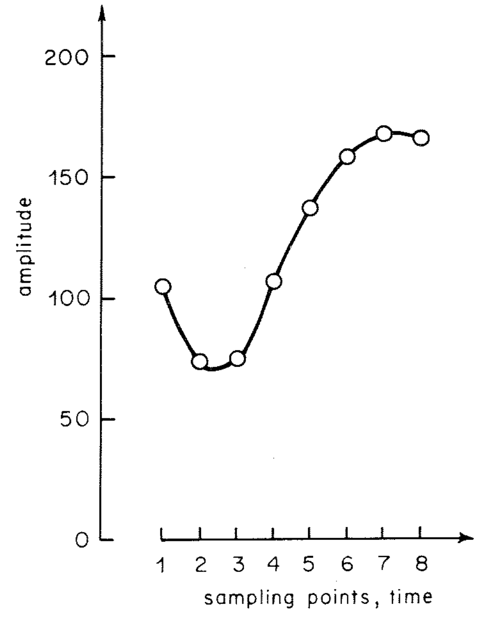

Fig. 3 - A portion of a television waveform showing sampling points

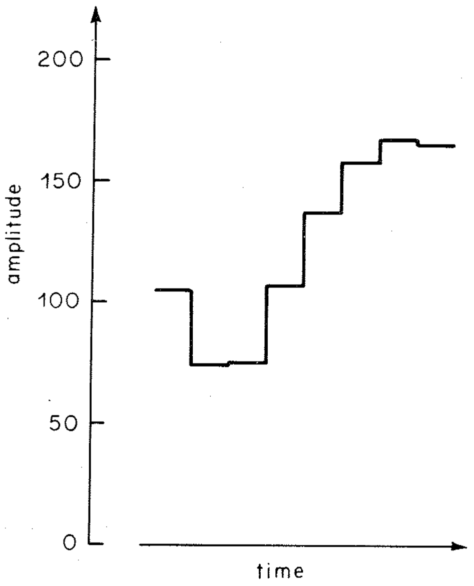

Fig. 4 - A sampled-and-held and quantised version of the waveform of Fig. 3

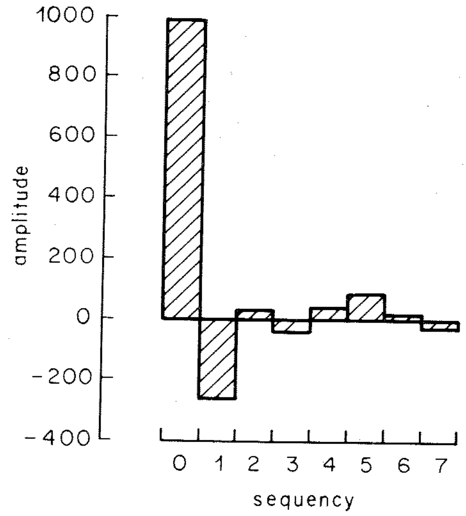

Fig. 5 - The sequency spectrum of the waveform of Fig. 4

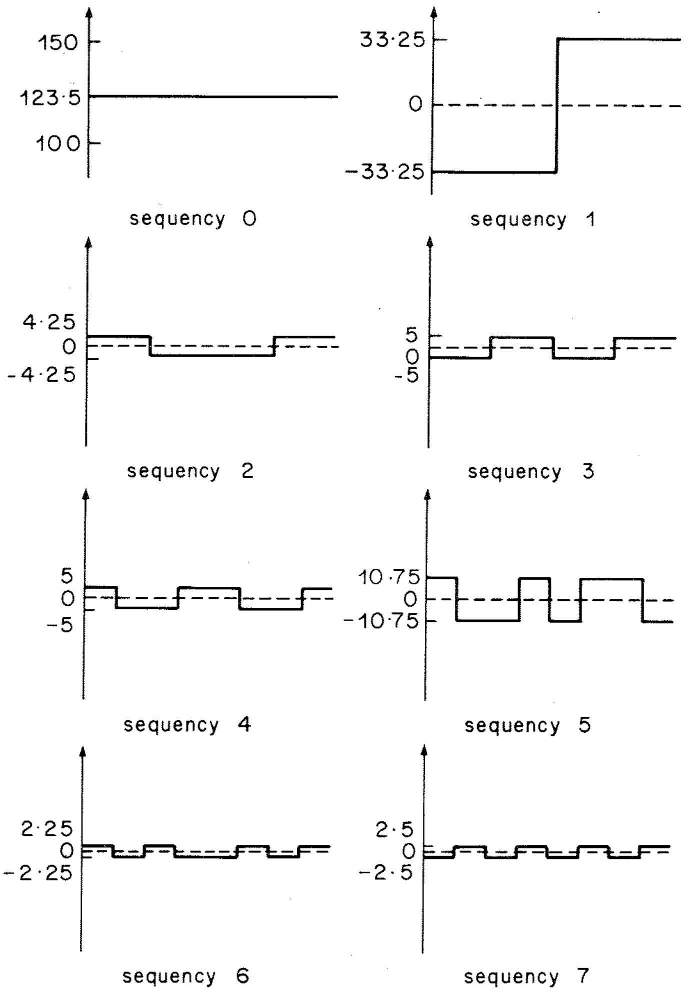

Fig. 6 - The set of Walsh functions into which the waveform of Fig. 4 can be separated

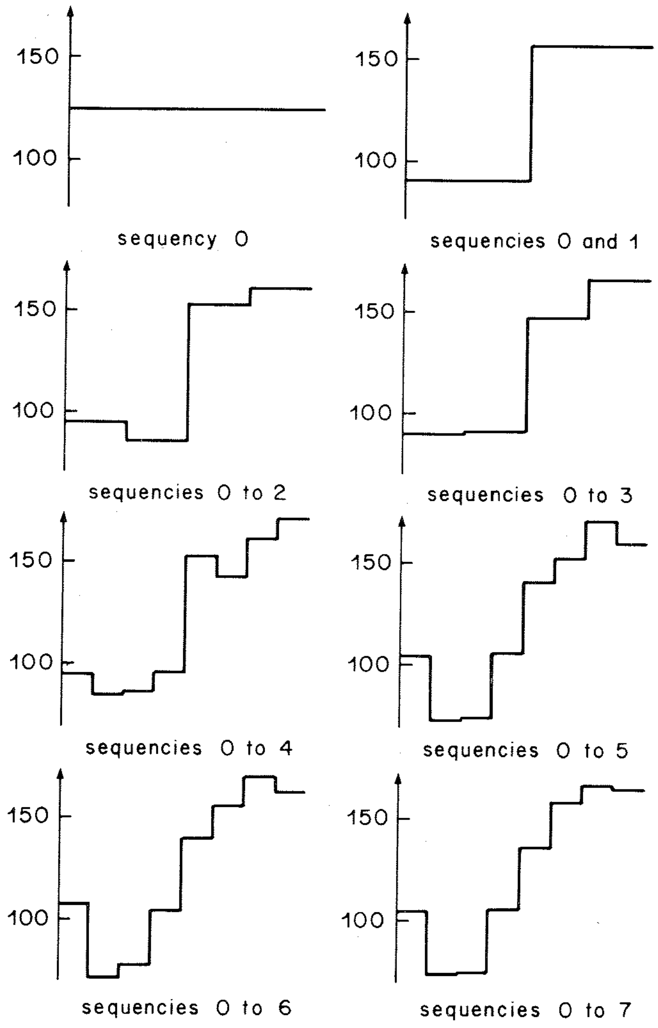

Fig. 7 - The effect of progressively adding together the Walsh functions of Fig. 6 to reproduce the original waveform

arranged to form the vector $\mathbf{x}$ in Equation (1). The simplest arrangement is to use n consecutive signal samples as the elements x_1 to x_n of the input vector. So, the set of eight samples shown in Table 3 may be written as the vector:

$$\mathbf{x} = \begin{bmatrix} 105 \\ 74 \\ 75 \\ 107 \\ 137 \\ 158 \\ 167 \\ 165 \end{bmatrix}$$

Multiplication of this vector by the (8 x 8) Walsh-Hadamard matrix involves the following set of operations and produces the vector of correlation coefficients shown:

$$\mathbf{c} = \begin{bmatrix} 105 + 74 + 75 + 107 + 137 + 158 + 167 + 165 \\ 105 - 74 + 75 - 107 + 137 - 158 + 167 - 165 \\ 105 + 74 - 75 - 107 + 137 + 158 - 167 - 165 \\ 105 - 74 - 75 + 107 + 137 - 158 - 167 + 165 \\ 105 + 74 + 75 + 107 - 137 - 158 - 167 - 165 \\ 105 - 74 + 75 - 107 - 137 + 158 - 167 + 165 \\ 105 + 74 - 75 - 107 - 137 - 158 + 167 + 165 \\ 105 - 74 - 75 + 107 - 137 + 158 + 167 - 165 \end{bmatrix} = \begin{bmatrix} 988 \\ -20 \\ -40 \\ 40 \\ -266 \\ 18 \\ 34 \\ 86 \end{bmatrix}$$

These operations are equivalent to multiplying the input signal by each of the Walsh functions shown in Fig. 1 and summing the resulting terms. Consequently, the sequencies to which the coefficients correspond are in the same order as the Walsh functions of Fig. 1. The sequency spectrum of the signal, shown in Fig. 5, is produced by arranging the coefficients in order of increasing sequency.

To recover the signal vector, the vector of coefficients is multiplied by the Walsh-Hadamard matrix and divided by a scale factor.

$$\mathbf{x} = \frac{1}{8} \begin{bmatrix} 988 - 20 - 40 + 40 - 266 + 18 + 34 + 86 \\ 988 + 20 - 40 - 40 - 266 - 18 + 34 - 86 \\ 988 - 20 + 40 - 40 - 266 + 18 - 34 - 86 \\ 988 + 20 + 40 + 40 - 266 - 18 - 34 + 86 \\ 988 - 20 - 40 + 40 + 266 - 18 - 34 - 86 \\ 988 + 20 - 40 - 40 + 266 + 18 - 34 + 86 \\ 988 - 20 + 40 - 40 + 266 - 18 + 34 + 86 \\ 988 + 20 + 40 + 40 + 266 + 18 + 34 - 86 \end{bmatrix} = \begin{bmatrix} 105 \\ 74 \\ 75 \\ 107 \\ 137 \\ 158 \\ 167 \\ 165 \end{bmatrix}$$

This inverse transformation in equivalent to an algebraic summation of eight Walsh functions, with amplitudes given by the transform coefficients divided by 8. Fig. 6 shows Walsh functions with amplitudes which are one-eighth the values given in the sequency spectrum (Fig. 5), and Fig. 7 shows the effect of progressively adding together the functions of Fig. 6 to reconstruct the signal.

2.3. The fast transform algorithm

By using a fast algorithm, the Walsh-Hadamard transform can be calculated more efficiently than by the direct matrix multiplication demonstrated in the previous section. In the fast transform method,^{4} the input vector is multiplied by a number of sparse matrices (matrices with a small number of non-zero elements), the product of which is the Walsh-Hadamard matrix:

$$\mathbf{H}_{2^p} = \mathbf{S}_p \mathbf{S}_{p-1} \dots \mathbf{S}_1$$

where each $\mathbf{S}$ is a (2^p x 2^p) matrix. Such a set of sparse matrices is formed by the following direct products:

$$\begin{aligned} \mathbf{S}_1 &= \mathbf{H}_2 \otimes \mathbf{I} \otimes \dots \otimes \mathbf{I} \\ \mathbf{S}_2 &= \mathbf{I} \otimes \mathbf{H}_2 \otimes \dots \otimes \mathbf{I} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \\ \mathbf{S}_p &= \mathbf{I} \otimes \mathbf{I} \otimes \dots \otimes \mathbf{H}_2 \end{aligned}$$

where $\mathbf{I}$ is the (2 x 2) unit matrix and $\mathbf{H}_2$ is the (2 x 2) Walsh-Hadamard matrix. For example, the (8 x 8) Walsh-Hadamard matrix can be expressed as a product of three sparse matrices $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$:

$$\mathbf{H}_8 = \mathbf{S}_3 \mathbf{S}_2 \mathbf{S}_1$$

$$\text{where } \mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{S}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

$$\mathbf{S}_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The transform is calculated in p stages, corresponding to the equations:

$$\left.\begin{array}{ll} \mathbf{a}_1 & = \mathbf{S}_1\mathbf{x} \\ \mathbf{a}_2 & = \mathbf{S}_2\mathbf{a}_1 \\ \cdot & \quad \cdot \\ \cdot & \quad \cdot \\ \cdot & \quad \cdot \\ \mathbf{a}_{p-1} & = \mathbf{S}_{p-1}\mathbf{a}_{p-2} \\ \mathbf{c} & = \mathbf{S}_p\mathbf{a}_{p-1} \end{array}\right\} \qquad (6)$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{p-1}$ are vectors produced at intermediate stages in the transformation. At each stage, an n-dimensional vector is multiplied by an $(n \times n)$ sparse matrix. Each sparse matrix has only two non-zero elements in each row. In half the rows both non-zero elements are 1's and, in the other half, one non-zero element is 1 and the other -1, so each product of a vector and a matrix requires only $\tfrac{1}{2}n$ additions and $\tfrac{1}{2}n$ subtractions. There are p such products in the transformation; therefore, the transform can be calculated in $np = n\log_2 n$ operations. For the same order of matrix the direct multiplication method requires $n - 1$ additions or subtractions to calculate each of the n coefficients, so the transform is calculated in $n(n - 1)$ operations. For a transformation of order 8 the fast algorithm uses only 24 operations, instead of the 56 operations required for the method shown in the previous section.

In practice, the range of numbers which can be produced at each stage is important, as this determines the number of binary processing elements needed to perform the transformation. The result formed by adding or subtracting two m-digit binary numbers contains $m + 1$ significant bits, so that the number of bits allocated in a fast transform processor must increase by one at each stage. Therefore, for a transformation of order 2^p, if m bits are used to represent the range of possible values of the input signal, $m + p$ bits must be used to represent the range of possible values of the transform coefficients.

It has been shown in Equation (4) that the inverse transformation for symmetric Hadamard matrices involves the same matrix multiplication as the forward transformation. However, the most economical method of fast inverse transformation uses multiplication by the set of sparse matrices in the reverse order to that used for the forward transformation. This minimises the number of bits needed at each stage in the inverse transformation.

For the fast inverse transformation, then, the Walsh-Hadamard matrix is expressed as the following product of the sparse matrices:

$$\mathbf{H}_{2^p} = \mathbf{S}_1\mathbf{S}_2 \ldots \mathbf{S}_p$$

and the inverse transformation consists of the following products:

$$\begin{array}{l} \mathbf{b}_1 = \mathbf{S}_p\mathbf{c} \\ \mathbf{b}_2 = \mathbf{S}_{p-1}\mathbf{b}_1 \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \mathbf{b}_p = \mathbf{S}_1\mathbf{b}_{p-1} \end{array}$$

where $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p$ represent the vectors produced at intermediate stages in the inverse transformation. Substituting Equations (6) gives the following set of equations relating the vectors produced at intermediate stages in the forward and inverse transformations:

$$\left.\begin{array}{l} \mathbf{b}_1 = \mathbf{S}_p\mathbf{S}_p\mathbf{a}_{p-1} \\ \mathbf{b}_2 = \mathbf{S}_{p-1}\mathbf{S}_p\mathbf{S}_p\mathbf{S}_{p-1}\mathbf{a}_{p-2} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \mathbf{b}_p = \mathbf{S}_1\mathbf{S}_2 \ldots \mathbf{S}_p\mathbf{S}_p\mathbf{S}_{p-1} \ldots \mathbf{S}_1\mathbf{x} \end{array}\right\} \qquad (7)$$

Since $\qquad \mathbf{S}_r\mathbf{S}_r = 2\mathbf{I} \qquad r = 1, 2, \ldots, p$

Equations (7) become

$$\begin{array}{l} \mathbf{b}_1 = 2\mathbf{a}_{p-1} \\ \mathbf{b}_2 = 2^2\mathbf{a}_{p-2} \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \mathbf{b}_p = 2^p\mathbf{x} \end{array}$$

Because the intermediate results are related by integer powers of two to intermediate results in the forward transformation, equal numbers of bits are required to describe the values in the inverse transformation to those required in the forward transformation. In general, the same number of bits are required in $\mathbf{b}_r$ as in $\mathbf{a}_{p-r}$, which requires $m + p - r$ bits. However, equivalent bits are r bits higher in significance in the inverse transformation because of the scale factor 2^r. This arises because the Hadamard matrix is not normalised (this is shown by the factor n in Equation (2)). Therefore, $\mathbf{b}_p$, the value produced at the output of

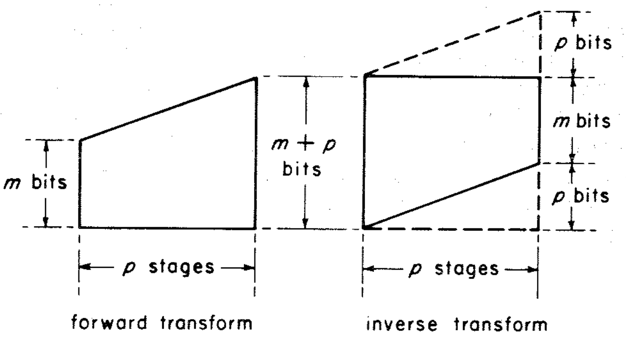

Fig. 8 - Allocation of significant bits in a transform system ($p = \log_2 n$)

the inverse transformer, must be divided by 2^p to regain the exact value of **X**, the input vector. In binary arithmetic this scaling operation is accomplished simply by relabelling the binary digits with lower significance.

The changes in the number of bits required at each stage of a transformation and inverse transformation of order n with an m-bit input are summarised in Fig. 8. For the inverse transformation two versions are compared: the continuous outline shows the number of bits needed to reconstruct the output of the forward transformation; the dashed outline shows the extra bits required for any combination of input values in the inverse transformation.

2.4. Practical Walsh-Hadamard transformers

The sparse matrices of the fast algorithm define the set of operations required to perform the transformation. In each case, multiplication of a vector by a sparse matrix involves the addition and subtraction of elements spaced a fixed number of elements apart in the vector. With consecutive signal samples used as the elements of the input vector, this operation corresponds to the addition to and subtraction from the signal of a delayed version of the signal. The arrangement of an adder, a subtractor, a shift register, and two data selectors shown in Fig. 9* will perform these calculations.[5] To transform blocks of 2^p samples, p such stages are required corresponding to the p Equations (6).

In Fig. 9 the shift register is used in two ways. First it delays some of the words representing the input samples so that words a fixed number apart in the signal can be added and subtracted. The sums are directed to the output, while the shift register, in its second use, delays the differences so that they can be fed to the output after the sums. Thus, the outputs are produced in an order consistent with the rows of the sparse matrix.

The same arrangement can be used for all the stages of both forward and inverse transformers. The stages differ only in the length of their shift registers, the control of the data selectors, and the word-length. In general, to transform n words each of m bits requires p transformer stages where $p = \log_2 n$. In the rth stage, the data selectors work with $(m + r)$ bits, the adder and the subtractor each accept inputs of $(m + r - 1)$ bits and produce outputs of $(m + r)$ bits, and the shift register contains $2^{(p-r)}$ words each of $(m + r)$ bits.

Therefore, transformations of larger blocks of data require approximately twice as many shift register elements with each unit increase in p. For large transformers ($p>5$), this approximately doubles the cost for each unit increase in p, so that distinct advantages must be gained from increases in the size of the transformation if the resulting large increases in cost are to be justified.

3. Transform domain bit-rate reduction

Any digital transformation which involves arithmetic operations inevitably produces a larger number of bits in the words describing the transform than were used in the input words. It has been shown that $m + p$ bits are required to represent each coefficient of the Walsh-Hadamard transform precisely, where the input words contain m bits and the order of the transformation is $n = 2^p$. It should be recognised, however, that this representation involves redundancy, since only 2^{mn} different sets of coefficients can be produced from the input words whereas $2^{(m+p)n}$ different sets of coefficients could be represented by n words each of $(m + p)$ bits.

With each increase in the order of the transformation, the number of bits needed to describe each coefficient of the transform increases by one. According to Parseval's theorem,[6] energy is conserved in an orthonormal transformation. So, to equalise the amounts of energy represented by the input signal and the transformed signal, the transform coefficients must be divided by the normalising factor $n^{1/2}$. This represents a shift of significance of the binary digits of the coefficients downwards by $\frac{1}{2}\log_2 n$ bits. Therefore, half the increase in the number of bits is used to represent greater amounts of energy and the other half to represent smaller amounts of energy than can be described by one of the input words. In Fig. 10 the binary digits representing the coefficients for different sizes of transformation are depicted as boxes; digits representing equal amounts of energy are shown at the same horizontal level. The diagram shows that a considerable reduction in the bit-rate of the transformed signal must be made even to achieve the bit-rate of the input signal.

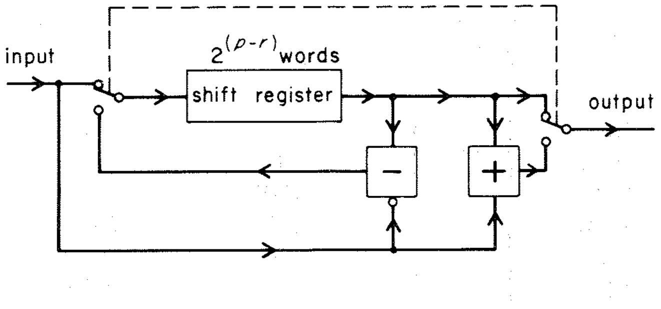

Fig. 9 - The rth stage of a Walsh-Hadamard transformer

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* This arrangement was devised by J.P. Chambers.

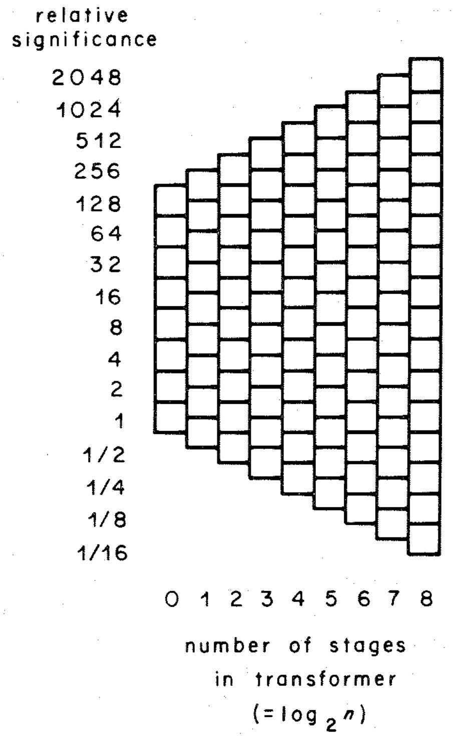

Fig. 10 - Relative significance of bits in the transform domain for different orders of transformation

The bit-rate of the transformed signal may in practice be reduced without impairing signal quality by omitting the most significant bits of some of the coefficients. This is possible because, owing to bandwidth and amplitude limitations, television signals are not able to produce sufficient amplitudes of some Walsh functions to use the full range of values provided by the corresponding coefficients. However, the saving to be made by excluding only these signals is thought to be negligible. Thus, to make any worthwhile saving, a large number of legitimate but improbable television signals must be excluded. When savings of this type have been made, it becomes possible for a coefficient to exceed the range allocated to it. If this occurs, it causes an error which appears in the picture as a Walsh function, the amplitude of which is the difference between the actual value and the transmitted value of the coefficient. Consequently, the magnitude of the error can be minimised by sending the permissible value nearest to the actual value; in practice, this is achieved by limiting.

Alone, this method is not sufficient to reduce the bit-rate of the transformed signal below that of the input signal. It can, however, be combined with other methods of bit saving.

The omission of the least significant bits of the coefficients makes a further saving. This introduces quantising errors which produce in the picture an interference signal formed by the combination of very low-level Walsh functions. Although such distortion is picture-dependent, it can be made to appear as noise. Once again, the errors can be minimised by sending the permissible value nearest to the actual value and this is achieved by rounding the coefficients to the level of the least significant bit transmitted.

The number of least significant bits that can be omitted can be estimated by considering the amounts of energy represented by the bits before and after transformation (Fig. 10). $\tfrac{1}{2}\log_2 n$ bits of each coefficient represent amounts of energy smaller than that described by the least significant bit of the 8-bit p.c.m. input signal. So omitting these bits in the transform domain will introduce into the output signal an average noise power approximately equivalent to that due to 8-bit p.c.m. quantisation.

The least significant bits of the transform coefficients are not independent. Because of this it is possible to omit selected least-significant bits from the transform coefficients to a total of $\tfrac{1}{2}n \log_2 n$, and then to re-insert the correct values of these bits in the inverse transformation. This method gives no increase in noise level due to the truncation of the transform coefficients. The method is described in the Appendix.

Other methods such as non-linear coding of coefficients and adaptive systems can be used to reduce the bit-rate of the transform further, but the processing involved in these methods is considerably more complicated.

Non-linear coding reduces the bit-rate by eliminating some of the code-states describing high amplitude coefficients; if these states are produced in the transform, then the nearest permitted state is used instead. Thus, small amplitudes are transmitted accurately, while some large amplitudes are transmitted with a small error. To work successfully, the method requires that small amplitudes should occur much more frequently than large amplitudes. Table 4 demonstrates how a 4-bit code can be compressed to a 3-bit code for transmission and then expanded back to a 4-bit code with only small errors in some of the high values.

TABLE 4

An Example of a Non-Linear Quantising Law for Bit-Rate Reduction

4-bit Code	Compressed Code	Expanded Code	Error Magnitude
0000	000	0000	0000
0001	001	0001	0000
0010	010	0010	0000
0011	011	0011	0000
0100	100	0100	0000
0101		0100	0001
0110	101	0110	0000
0111		0110	0001
1000	110	1001	0001
1001		1001	0000
1010		1001	0001
1011		1001	0010
1100	111	1101	0001
1101		1101	0000
1110		1101	0001
1111		1101	0010

Some adaptive systems attempt to exploit the property that several large values cannot occur together in a set of coefficients. This is because a single large coefficient may convey almost all the energy in a block of signal samples, and therefore, since energy is conserved in the normalised transformation, all other coefficients must be small. A number of adaptive systems have been considered in which a large number of bits are used to describe a coefficient only when that coefficient has a large value. The most significant bits of the remaining coefficients are then omitted. However, this saving is partly offset by extra bits needed to identify the large coefficients. Because of this, the overall saving is not likely to be large enough to justify the complexity of the processing equipment required.

Several other methods of bit-rate reduction have been used successfully with the high-order transformations of pictorial data performed on digital computers.[7,8] In general, these methods cannot be applied to broadcast quality p.c.m. television signals because, at the data rate used, the processing equipment would be prohibitively expensive.

4. A Walsh spectrum analyser

4.1. Spectral analysis with Walsh functions

The performance of the simpler methods of bit-rate reduction, in which some of the most and least significant bits of the coefficients are omitted, is readily predictable. The noise introduced by omitting a number of the least significant bits was estimated in the previous section, and the number of most significant bits that can be removed can be found by measuring the maximum values of the transform coefficients produced by television signals.

Walsh-Hadamard transformation is equivalent to correlation of the signal with the set of Walsh functions (Equation (5)). To transform real-time television by direct correlation would require n correlators fed by a generator producing all of the set of n Walsh functions simultaneously. However, television signals may be transformed in slow time using a spectrum analyser consisting of a single correlator and a generator producing Walsh functions one at a time. The transform is then found by feeding the signal n times to the correlator, each time with a different Walsh function, to produce the n coefficients.

Since only the distribution of amplitudes in the spectrum is required in the prediction of bit rates, it is not necessary to calculate all of the coefficients for each portion of signal. It is more convenient to correlate the signal directly with Walsh functions generated one after another. As each Walsh function is correlated with a different portion of the signal, the sets of coefficients produced do not constitute a full transformation of the signal. However, provided the picture period is not simply related to the Walsh function period, over several picture periods every area of the picture will be analysed with every Walsh function. Therefore, for a still picture, all values of the coefficients which can be produced by that picture will be produced. Thus the maximum amplitude of each spectral coefficient is measured.

The slow-time analysis method has the advantage that comparative measurements for different orders of transformation can be made merely by altering the period of the Walsh functions. Also, by suitably controlling the generator, the functions can be correlated with the signals corresponding to rectangular areas of the picture, so that the benefits of two-dimensional transformations may be assessed. The coefficients of any other Hadamard transformation could similarly be found by replacing the Walsh function generator by a generator which produces the set of basis functions of the required transformation.

The Walsh spectrum analyser thus consists of a Walsh function generator, a correlator and a maximum-value store. Digital arithmetic and storage are used, and a digital-to-analogue converter is used to allow the stored values to be displayed on an oscilloscope. A block diagram of the equipment is shown in Fig. 11.

4.2. Walsh function generation

Since the product of two Walsh functions is itself a Walsh function, it is possible to choose subsets of p functions which, when multiplied together, produce the complete set of 2^p functions. In general, multiplication of Walsh functions with sequencies i and j produces the function of sequency $i \oplus j$, where $\oplus$ denotes modulo-two addition.* It is convenient to choose, as the functions of the subset, functions of sequency 1 (or as a binary

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* Modulo-two addition: the exclusive-OR function of the binary digits of two numbers.

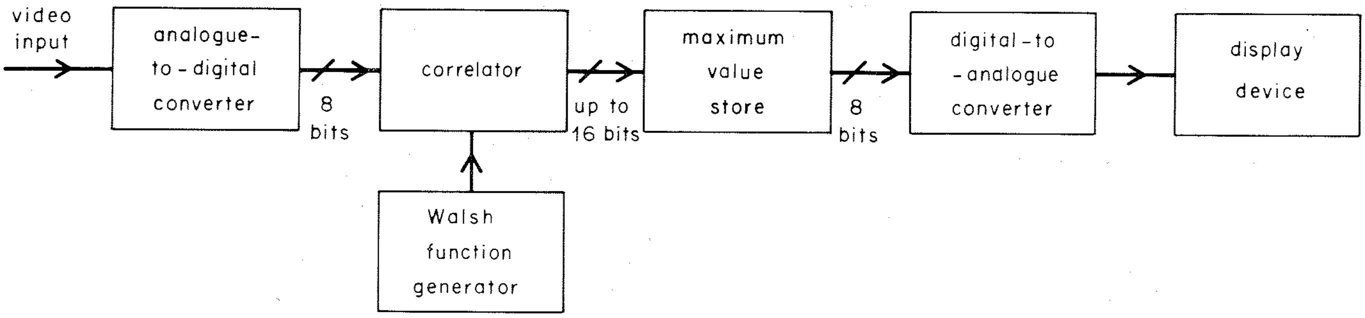

Fig. 11 - Block diagram of the Walsh spectrum analyser

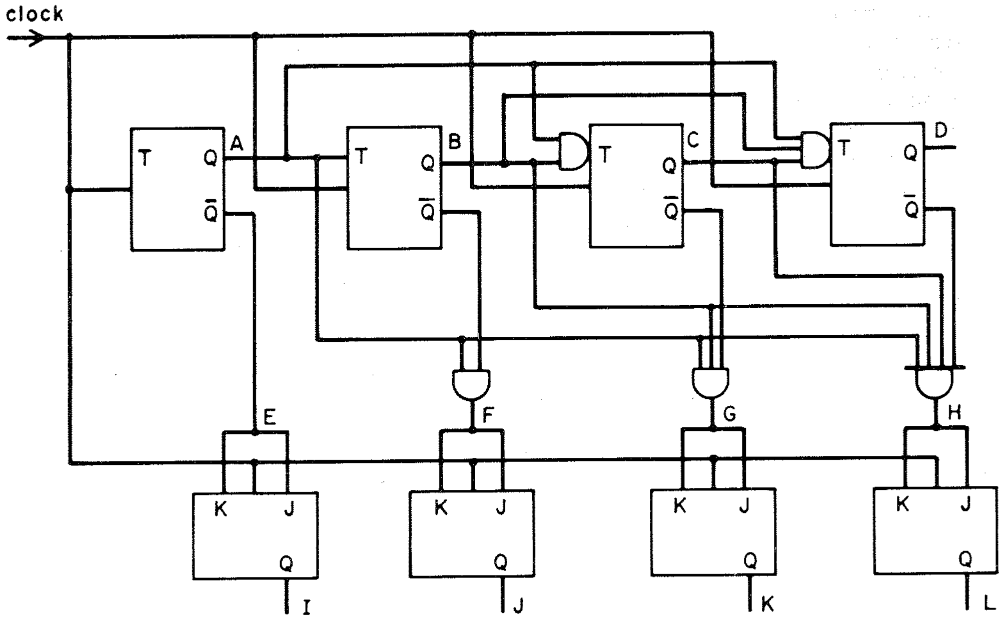

Fig. 12 - Generation of a subset of Walsh functions

number 00. . . .0001), 2 (00. . . .0010), 4 (00. . . .0100), 8 (00. . . .1000), ..., that is, 2^{r-1} where $r = 1, 2, \ldots, p$. Then, the functions to be multiplied together to form a particular function are indicated by the positions of '1's in the binary representation of the sequency. For example, to produce a Walsh function of sequency 13 (00. . . .1101) the functions with sequencies (00. . . .1000), (00. . . .0100) and (00. . . .0001) must be multiplied together.

In logic circuits, the two values of Walsh functions, +1 and −1, can be represented conveniently by 0 and 1 respectively. Then the Walsh functions of the subset may be generated from a binary counter by decoding states in which a '0' is followed by less significant '1's and using these signals to change the states of J–K bistables. A circuit for generating a subset of four Walsh functions is shown in Fig. 12 and the waveforms generated are shown in Fig. 13. The waveforms in Fig. 13(*c*) comprise part of the required set of Walsh functions. With +1 and −1 represented by 0 and 1, multiplication is replaced by the exclusive-OR operation. Therefore, the other Walsh functions of the set can be produced by combining the functions of the subset in exclusive-OR gates. However, as only one function is required at a time for Walsh spectral analysis, the circuit of Fig. 14 may be used. This arrangement is particularly convenient because the boxed part of the circuit is known as a binary rate multiplier and a six-bit version is available as an integrated circuit.* The sequency of the function to be generated is applied in binary form to the AND-gate inputs A, B, C and D to enable the state of the J–K bistable to be changed at the appropriate transition points. Higher sequency functions may be generated by extending the binary counter and adding extra AND-gates. By driving the AND-gate inputs from a binary counter advanced by one at the end of each function, a

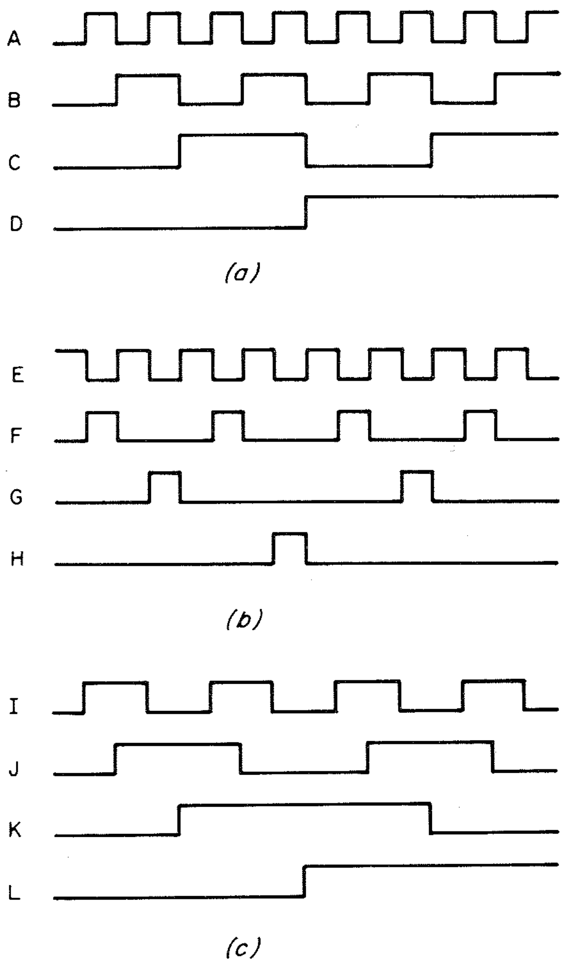

Fig. 13 - Waveforms in Fig. 12
(*a*) Outputs of the binary counter
(*b*) Outputs of the decoding gates
(*c*) Outputs of the J–K bistables

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* The use of this integrated circuit in Walsh function generation was suggested by J.P. Chambers.

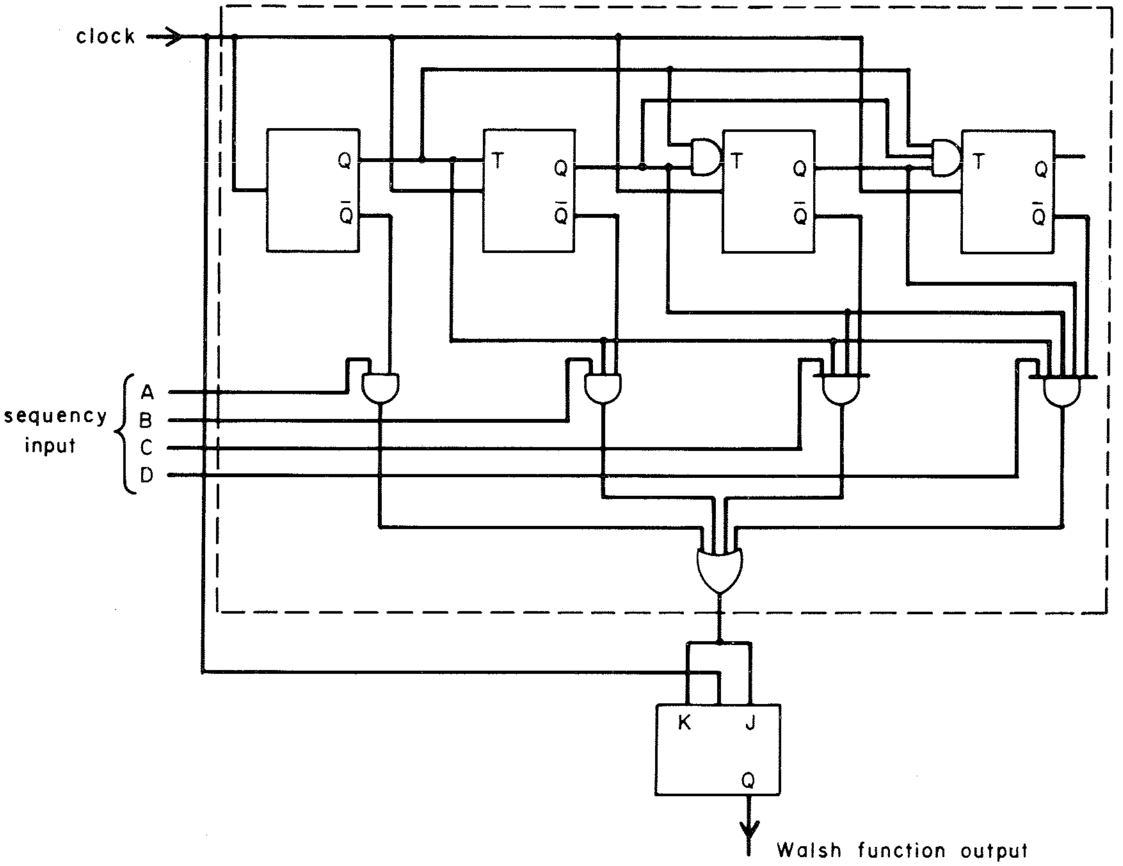

Fig. 14 - An alternative Walsh function generator

complete set of Walsh functions may be generated, one after another with increasing sequency.

This provides the basis of the Walsh function generator used in the spectrum analyser. The 'sequency sweep' produced by the generator and fed to the correlator is analogous to the output of the swept-frequency oscillator used in Fourier spectrum analysers.

4.3. Correlation using Walsh functions

The correlation of two discrete functions is measured by summing the products of corresponding function values as shown in Equation (5). So, in general, a correlator consists of a multiplier to form the products, an adder to form the sums, and a store to hold intermediate values in the calculation.

Correlation is much simpler when one of the functions is a Walsh function. Then, as the Walsh function value can be only +1 or −1, multiplication can change only the polarity of the corresponding value in the other function. When digital television signals are correlated with Walsh functions, the 8-bit codewords representing the signal samples are passed unchanged to the adder if the corresponding Walsh function value is +1. If the Walsh function value is −1, then negative numbers having the same magnitude as the signal samples are passed to the adder.

The correlator in the Walsh spectrum analyser uses two's complement notation to represent negative numbers. The two's complement of a binary number is formed by complementing each digit of the number and adding one to the resulting word. Hence, the binary code is continued below zero with numbers as follows:

2	0 010
1	0 001
0	0 000
−1	1 111
−2	1 110
−3	1 101
−4	1 100
.	. .
.	. .
.	. .

This representation allows both positive and negative numbers to be added together using an ordinary binary adder.

After multiplication, the codewords describing the products with each Walsh function are summed in the accumulator (Fig. 15). This consists of an adder, the output of which is fed to a store. The output of the store is fed to one input of the adder and the products are fed to the other input. Each product is added to the sum of all the previous products and the result stored. When all the products corresponding to one Walsh function have been added together, the result is a transform coefficient.

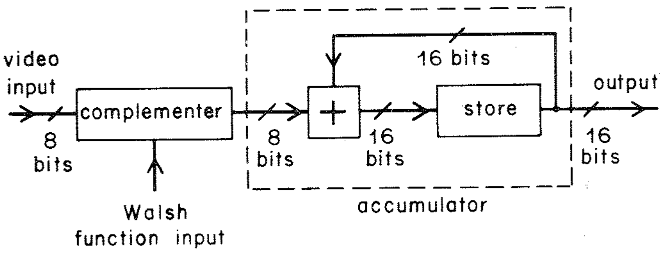

Fig. 15 - The correlator used in the Walsh spectrum analyser

Because the transform coefficients can have a larger dynamic range than the input samples, 16-bit arithmetic is used in the correlator. This allows the coefficients of transformations of up to 256 8-bit words to be calculated.

4.4. Displays of sequency spectra

The sequency spectra measured by the analyser can be shown in several different ways depending on how the sequency coefficients are processed before digital-to-analogue conversion. If the output of the correlator is converted directly to an analogue signal and fed to an oscilloscope, the coefficients are shown in sign and magnitude form to produce a display similar to Fig. 5. Also the sign may be suppressed to produce a display of magnitude against sequency.

The most important feature to be determined is the largest magnitude that each sequency coefficient attains, since this fixes the upper limit of the coding range for each coefficient and so determines the required bit-rate for linear coding. To facilitate the measurement of the maximum values, two other processing operations can be included before digital-to-analogue conversion. The first of these uses a comparator and a register to store the largest amplitude of each coefficient. Then the contents of the storage register can be displayed as a spectrum of maximum amplitudes. The second operation consists of finding the position of the most significant '1' in the binary number used to represent each amplitude. The position of the most significant '1' can be represented by a three-bit code, which, after conversion to analogue form, produces an eight-level signal. When displayed on an oscilloscope, this signal clearly identifies the magnitude of the most significant bit used by each coefficient; thus the number of unused bits may be found.

5. Sequency spectra of television signals

5.1. Factors affecting the spectra

The purpose of an orthogonal transformation in bit-rate reduction is to redistribute the energy of the signal between the transform coefficients. Then, if the distribution is favourable, the set of transform coefficients may be represented by fewer bits than the original signal. However, the way in which the energy is distributed depends not only on the transformation, but also on the sampling of the analogue signal and the way in which the samples are grouped together before transformation. The extent to which these factors may be manipulated to produce more favourable energy distributions will be described by examining the sequency spectra of a number of transformation systems.

First, Walsh-Hadamard transformation of 32 samples taken sequentially (as in Section 2.2) will be considered with a sampling frequency (f_s) of 13·296875 MHz. This frequency, a multiple of line frequency which is near to three-times the colour subcarrier frequency, ensures that all possible values of the spectral coefficients are measured. The spectrum obtained with this transformation system will be used to demonstrate the effect of changes in picture content. Subsequently, the effects of different sampling frequencies, different sample groupings and a different transformation will be described. Fortunately, the effects of these changes are generally independent of each other, so that they may be described separately.

For all the measurements, the analogue-to-digital converter was adjusted so that a 100% colour-bars signal[9] just filled the conversion range.

5.2. The spectra of video signals

It was shown in Section 2.3 that $m + \log_2 n$ bits were required to represent each coefficient of the transform of n m-bit numbers. However, because the analogue video signal is limited in bandwidth to 5·5 MHz and in amplitude to 0·7V between black-level and white-level,[9] the p.c.m. video signal cannot produce all combinations of n 8-bit numbers. In fact, some of the coefficients produced by transforming video signals can be represented accurately with fewer than $8 + \log_2 n$ bits.

The maximum possible value of each coefficient is produced when the video signal takes the form of an appropriately phased Walsh function corresponding to that coefficient with a magnitude of 0·7V peak-to-peak and limited in bandwidth to 5·5 MHz. For Walsh-Hadamard transformation of thirty-two samples, grouped sequentially and with a sampling frequency of 13·296875 MHz, the sequency spectrum of maximum possible values is shown in Fig. 16. The low-pass filtering has little effect on the low-sequency Walsh functions, whereas the amplitudes of the high-sequency functions are reduced considerably. The number of bits which are never used are shown in row A of Table 5. Although a total of 26 bits can be removed, this is much less than the $\frac{1}{2}n\log_2 n$ (in this case, 80) extra most significant bits which have to be added during the transformation. Consequently, to be able to transmit all possible television signals without any loss of accuracy, more bits must be used to describe the transform coefficients than were used to describe the original signal.

The amplitude limitation of 0·7V applies only to luminance signals and the coded colour signal can exceed this value. For example, the waveform representing saturated red has a peak-to-peak magnitude of 0·885V.[9] The spectrum of this waveform is shown in Fig. 17. Although the amplitude of the signal waveform is greater than 0·7V, with this sampling frequency the transform coefficients are all smaller than those of Fig. 16. This is

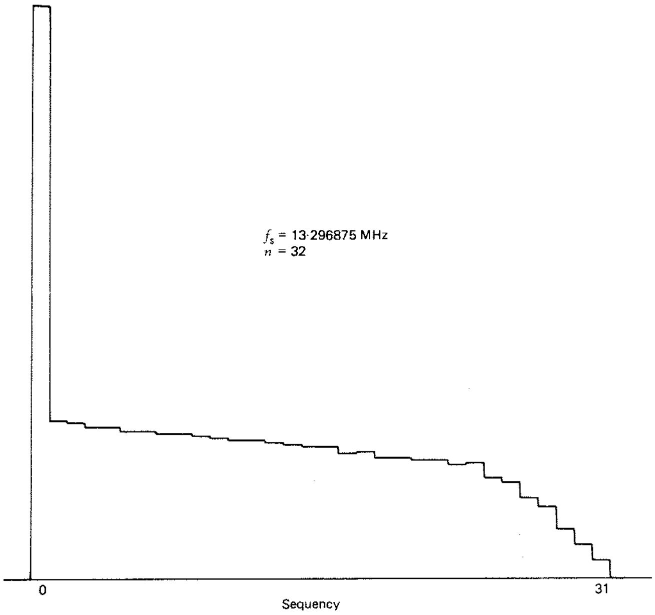

Fig. 16 - The sequency spectrum of 0·7V_{p-p} Walsh functions limited in bandwidth to 5·5 MHz*

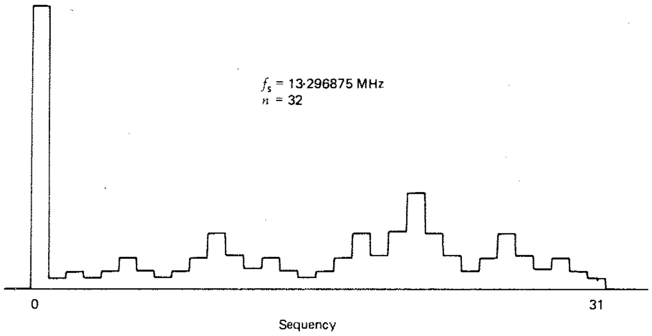

Fig. 17 - The sequency spectrum of a 0·885V_{p-p} colour subcarrier waveform

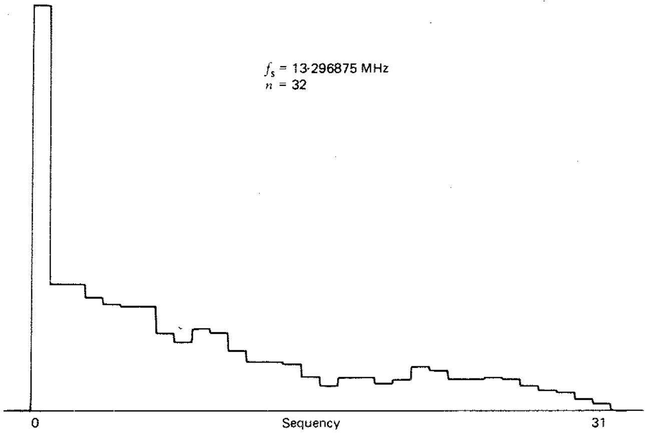

Fig. 18 - The sequency spectrum of a television signal representing white alphanumeric characters on a black background

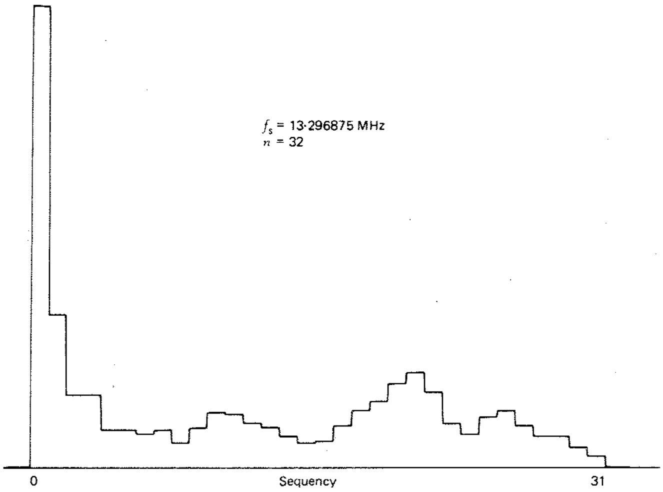

Fig. 19 - The sequency spectrum of a 100% colour bar waveform

* In this Figure and in Figs. 17, 18, 19, 20, 21, 23, 24, 26, 27, 29, 30, 33, 35, 38 and 39, the term sequency spectrum is used to describe a graph showing the largest magnitudes of the transform coefficients plotted against sequency. The graphs have been produced for signals representing still pictures by measuring the coefficients over many picture periods and retaining the largest value for each sequency. The polarities of the individual coefficients are ignored for these graphs, but could be positive or negative for all coefficients except the zero-sequency or average-level coefficient, which is always positive. The ordinate scales used are the same for graphs relating to the same transformation block size (n). For different block sizes the scales are related in the ratio of the block sizes; so, for example, magnitudes that appear as the same size in graphs for n = 32 and n = 64 are, in absolute terms, twice as large in the graph for n = 64.

These points also apply to the graphs shown in Figs. 42 and 43, although, because of the different transformation used, the abscissa is no longer a measure of sequency, but relates to the phasing of the pseudo-random sequence.

TABLE 5

The Numbers of Unused Most Significant Bits of Each Coefficient for the Spectra of Figs. 16, 18 and 19

Sequency	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	Total
A	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	3	26
B	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	2	2	3	2	2	2	2	2	2	2	3	3	3	4	5	62
C	0	0	1	1	2	2	2	2	3	2	2	2	2	2	3	3	3	2	2	2	1	1	1	2	2	2	2	2	3	3	3	4	64
D	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	2	2	2	1	1	1	2	2	2	2	2	3	3	3	4	54

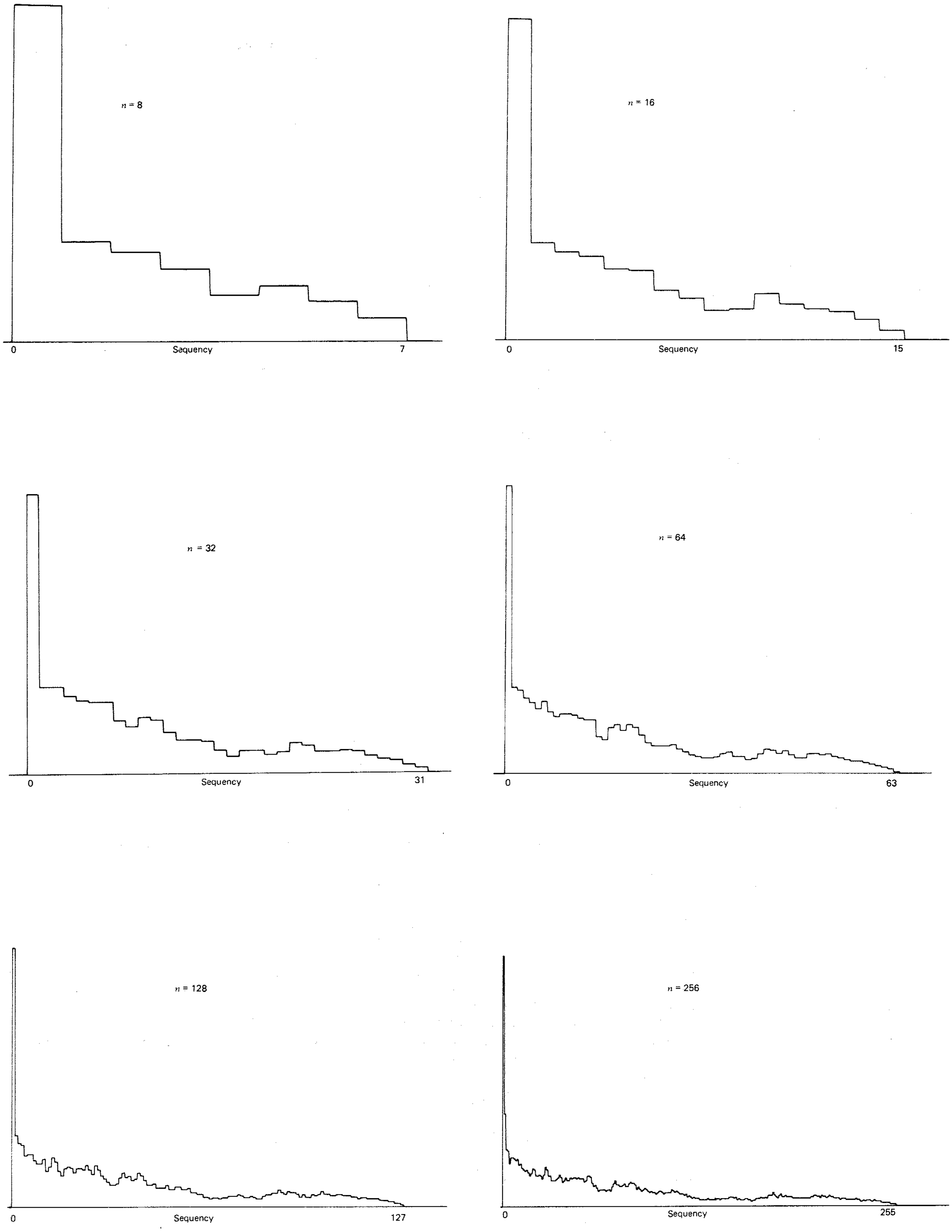

Fig. 20 - Sequency spectra of the alphanumeric characters signal for values of n from 8 to 256 (f_s = 13·296875 MHz)

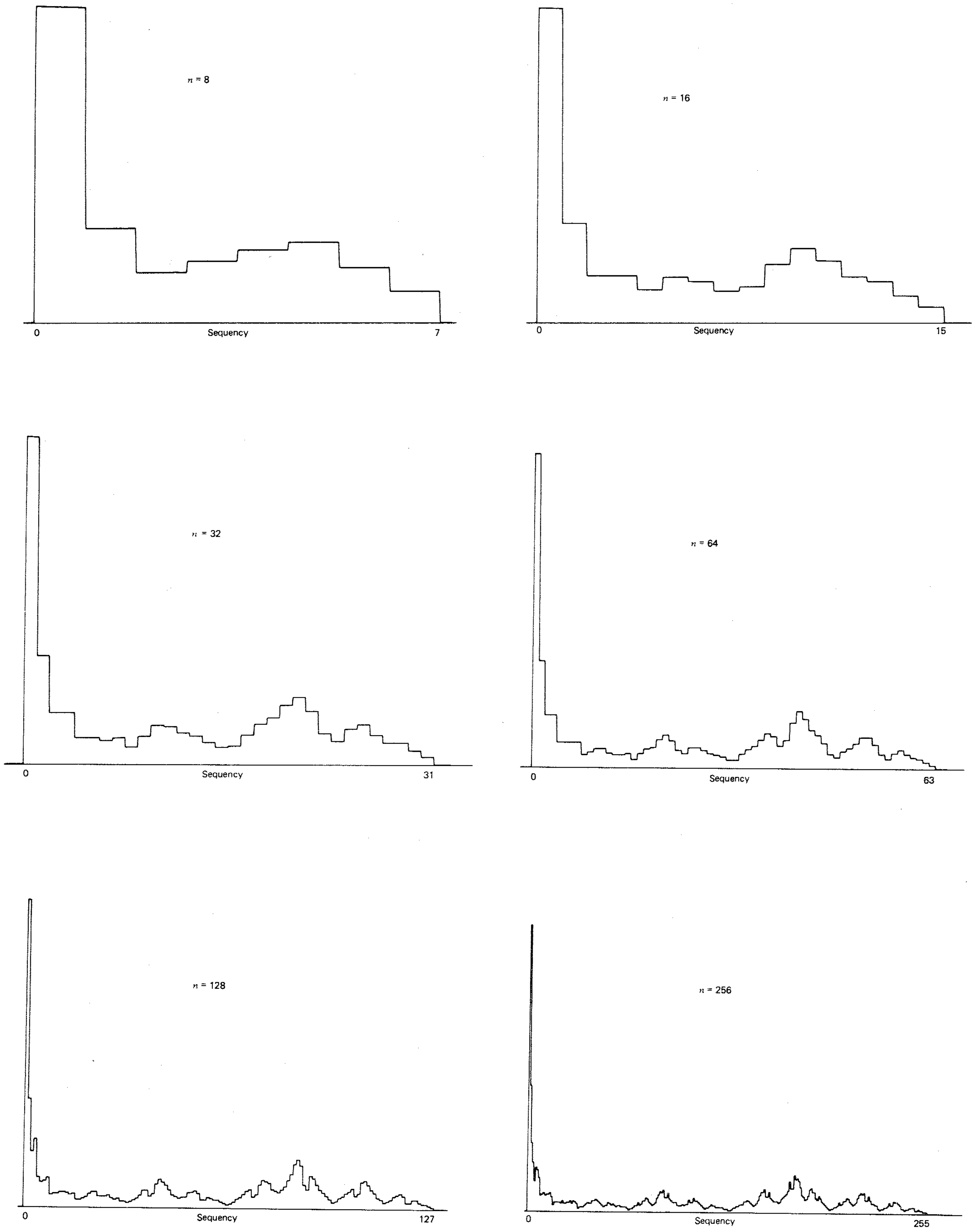

Fig. 21 - Sequency spectra of a 100% colour bar waveform for values of n from 8 to 256 (f_s = 13·296875 MHz)

because the energy of the sinusoidal subcarrier waveform is split between a number of coefficients.

In practice, the spectrum approaches the theoretical limits of Fig. 16 most closely for signals containing large numbers of fast transitions between black-level and white-level, and for signals containing high amplitudes of colour subcarrier. For example, Fig. 18 shows the sequency spectrum of a video signal representing a large number of white alphanumeric characters arranged in a regular format against a black background. Fig. 19 shows the sequency spectrum of the 100% colour bar waveform.[9] In general, the coefficients in the lower half of the sequency spectrum are larger in Fig. 18 than in Fig. 19, whereas the coefficients in the high-sequency half of the spectrum are larger in Fig. 19. By combining the results for these two signals, taking the larger value for each coefficient in each case, a spectrum is produced which is very seldom exceeded by television signals. The number of bits in each coefficient which are not used in these two signals are shown in Table 5; row B is for alphanumeric characters only, row C is for colour bars only, and row D is for the two signals together. It is convenient to use this combined spectrum as the basis for comparison of different systems, since this spectrum is representative of the most difficult that a bit-rate reduction system for normal broadcast television signals will meet.

Picture material which does not contain fast transitions between black and white, and which is not highly coloured produces a spectrum in which many coefficients are several times smaller than those of Figs. 18 and 19.

5.3. The effect of changes in the order of the transformation

The order of the transformation, denoted by n, the number of samples transformed, is an important variable. It is reasonable to expect that increasing n may lead to larger bit-rate reductions as this will allow the transformation to make use of the correlation between more widely-spaced samples. To what extent this is true for television signals can be found by examining the sequency spectra for different values of n.

Sequency spectra for transformations of order 2^3 to 2^8 samples taken sequentially are shown for alphanumeric character signals in Fig. 20 and for 100% colour bar signals in Fig. 21. For each signal, the shapes of the spectra are similar for all transform sizes. The higher resolution of the larger transformations reveals more detail in the structure of the spectra. The diagrams show that the coefficients decrease in amplitude relative to the zero-sequency coefficient as the transformation order increases.

To assess the bit-rate required for each value of n, the average number of unused most significant bits is subtracted from the $8 + \tfrac{1}{2}\log_2 n$ bits per coefficient required in theory. Fig. 22 shows the average number of bits per coefficient as a function of the order of the transformation (n) for a number of different signals. The graphs suggest that large transformations do not necessarily lead to a reduction in bit-rate since the range of coefficients with significant amplitudes tends to increase with n.

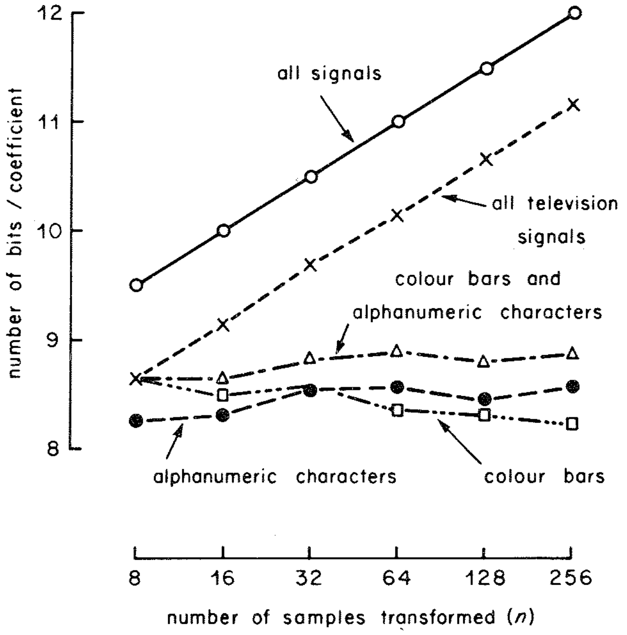

Fig. 22 - The variation of bit-rate with the order of the transformation

5.4. The effect of changes in sampling frequency

The sampling frequency used in the transformation system acts as a scaling factor between frequencies in the signal and the sequencies of the functions represented by the coefficients. The highest-sequency coefficient represents the amplitude of the half-sampling frequency component in the signal. Therefore, as the sampling frequency is reduced, both the highest-sequency coefficient and all the other coefficients carry information corresponding to lower frequencies in the signal. Since, in monochrome signals, the lower frequencies are usually present with higher amplitudes, lowering the sampling frequency tends to cause a small increase in the average number of bits required for each coefficient. Conversely, increasing the sampling frequency reduces slightly the average number of bits per coefficient.

When signals containing large amplitudes of colour subcarrier are considered, the effect of changes in sampling frequency become more important. This is because particular sampling frequencies concentrate the colour information into some transform coefficients. This occurs when there are whole numbers of cycles of the colour subcarrier in each block of samples for transformation. For a block of n samples, this happens when the sampling frequency is $^{n}/n_{c}$ times the subcarrier frequency, where n_c is an integer representing the number of cycles of subcarrier in the block. The sequency spectra of a subcarrier-frequency sine wave of 0·885V peak-to-peak magnitude are shown in Fig. 23 for $n = 32$ and $n_c = 10, 11, \ldots, 16$. None of these values produces a block rate having a simple relationship to line frequency. Over this range, as the sampling frequency is reduced, the sequency of the largest coefficient (excluding the zero-sequency coefficient) increases. Also, if n_c has factors in common with n, the number of coefficients which respond to subcarrier signals is reduced.

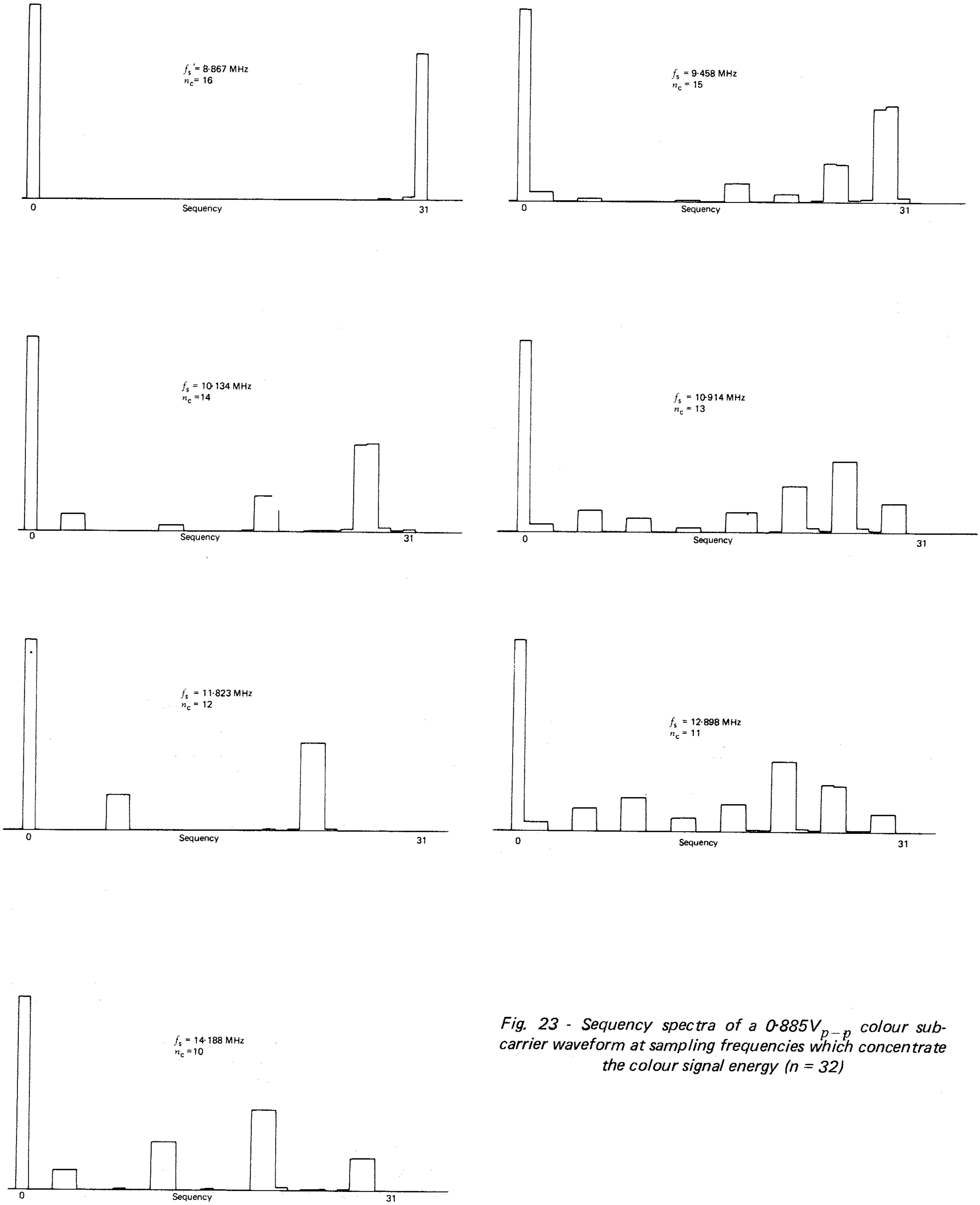

Fig. 23 - Sequency spectra of a $0{\cdot}885V_{p-p}$ colour sub-carrier waveform at sampling frequencies which concentrate the colour signal energy (n = 32)

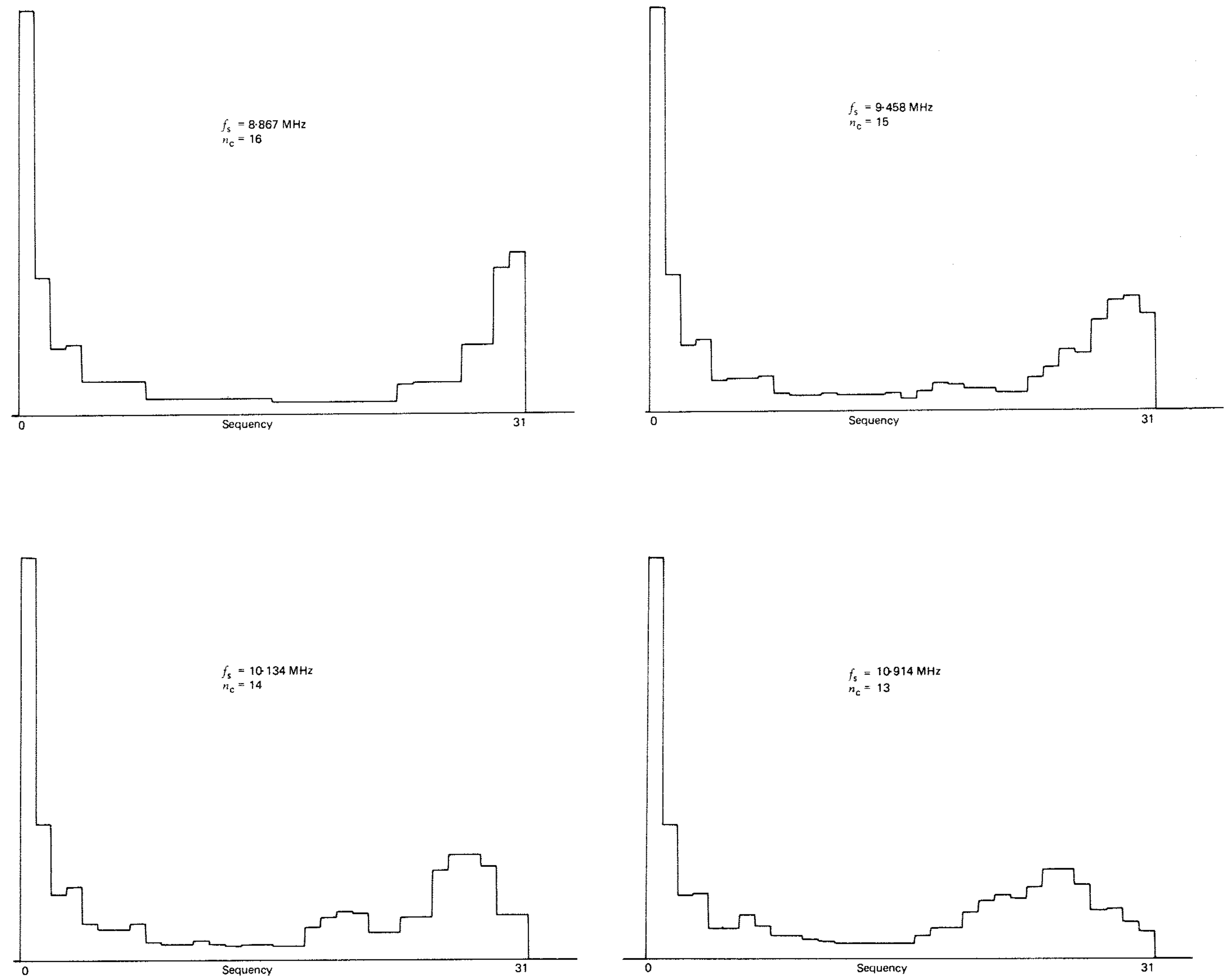

Fig. 24 - Sequency spectra of a 100% colour bar waveform at sampling frequencies which concentrate the colour signal energy (n = 32)

When the subcarrier is modulated by colour changes, sidebands are produced around the sequencies corresponding to subcarrier frequency. This is demonstrated by the sequency spectra of 100% colour bars which are shown in Fig. 24 for n = 32 and n_c = 10, 11, ..., 16. For n_c = 13 to 16 the sampling frequencies are below the Nyquist rate for a 5·5 MHz bandwidth signal.

The average number of bits required to represent each coefficient is shown in Fig. 25 for sampling frequencies ranging from twice the colour subcarrier frequency ($2f_{sc}$) to over three-times the colour subcarrier frequency ($3f_{sc}$). The measurements were made at frequencies for which n_c is an integer. For the colour bar signal, the graph shows a small advantage in using sampling frequencies such as $2f_{sc}$ and $(8/3)f_{sc}$ (n_c = 12) which concentrate the energy of the subcarrier into a small number of components. For other signals, a reduction in sampling frequency leads to a small increase in the number of bits per coefficient, as expected. When both types of signal are considered together, there is a slight increase in the number of bits per coefficient as the sampling frequency is reduced.

Larger values of n reveal more closely the structure of the sidebands caused by modulation of the subcarrier. Fig. 26 shows the spectra of 100% colour bars for a transformation of order 256 with sampling frequencies of $2f_{sc}$ and $(8/3)f_{sc}$ in comparison with that for $3f_{sc}$. Further, Fig. 27 shows the spectra for alphanumeric characters with n = 256 at the same sampling rates. This demonstrates the slight shift of energy to higher sequencies as the sampling frequency is reduced.

Changes in sampling frequency do not produce marked changes in the sequency spectra of signals containing only luminance information until block-lengths of more than one television line are used. This is because until then, in general, there is no periodic structure in the waveform which can correspond strongly to particular Walsh functions.

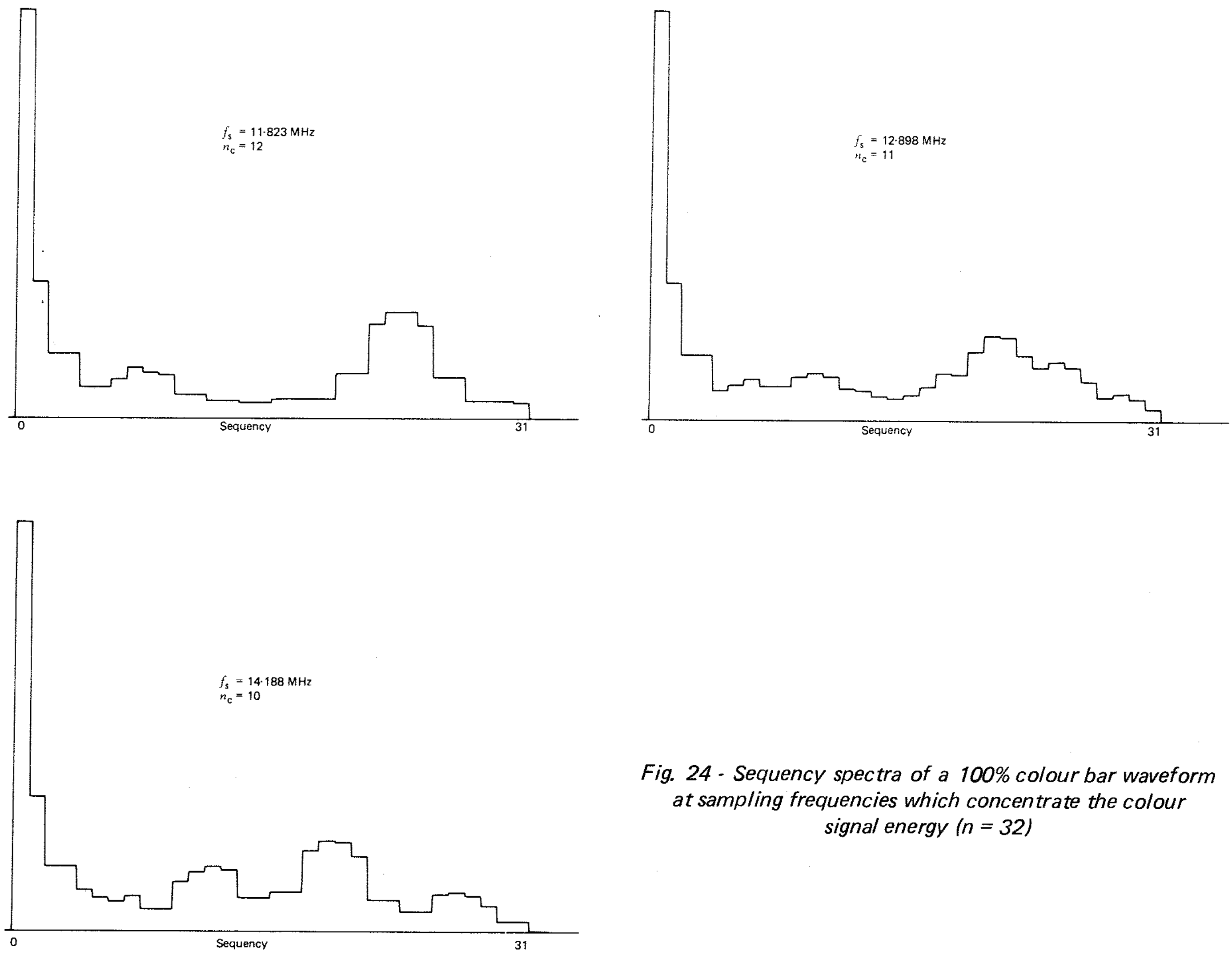

Fig. 24 - Sequency spectra of a 100% colour bar waveform at sampling frequencies which concentrate the colour signal energy (n = 32)

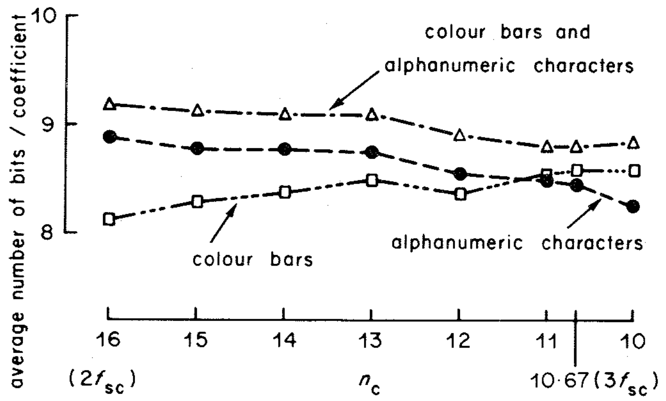

Fig. 25 - The variation of the average number of bits required to represent the coefficients at sampling frequencies which concentrate colour signals into a reduced number of coefficients (n = 32)

5.5. A method of concentrating colour signals by sub-sampling*

If the sampling frequency is an integer multiple of the colour subcarrier frequency, then it is possible to concentrate the colour information into the low-sequency transform coefficients by re-grouping the samples before transformation. For a sampling frequency of A-times subcarrier frequency (where A is an integer) the method consists of forming A groups each containing n regularly spaced samples taken at subcarrier-frequency rate. Therefore, in a section of the waveform representing an area of constant colour, all n samples in a group will be equal. When these groups of n samples are transformed, all the energy of the signal will be contained in the zero-sequency coefficients and the amplitudes of all other coefficients will be zero. When the subcarrier signal is modulated by colour changes, the associated sidebands are carried as low-sequency coefficients.

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* This method was suggested by Dr. G.J. Phillips.

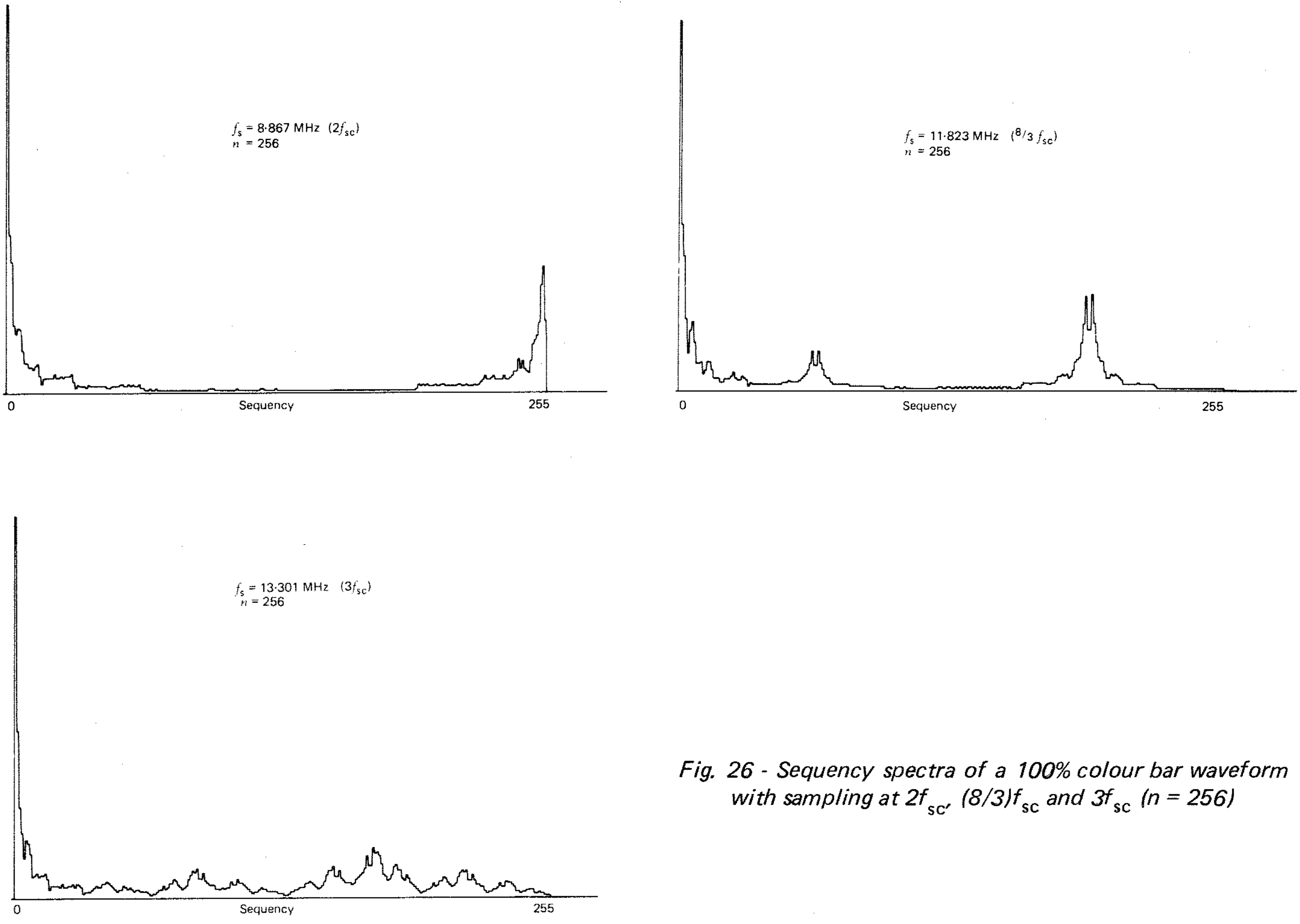

Fig. 26 - Sequency spectra of a 100% colour bar waveform with sampling at $2f_{sc}$, $(8/3)f_{sc}$ and $3f_{sc}$ (n = 256)

With this method, the majority of the energy from both luminance and chrominance signals is concentrated into the low-sequency coefficients. This can be shown by considering the frequency spectra of the sub-sampled blocks of signal. Fig. 28 shows the frequency spectrum of a video signal sampled at subcarrier rate compared with that for three-times subcarrier sampling. The baseband part of the sub-sampled signal, that is the portion between 0 and half-sampling frequency (in this case, half-subcarrier frequency), is shown at (*d*) in Fig. 28. It can be considered to contain part of the upper sideband of the baseband signal, part of the lower sideband of the signal centred on f_{sc}, and part of the lower sideband of the signal centred on $-f_{sc}$. Alternatively, it can be considered as a version of the video spectrum before sampling, folded back on itself at subcarrier frequency, and then folded back again at half-subcarrier frequency. When the sub-sampled signals are transformed, the low frequencies in the baseband spectrum of the sub-sampled signal (Fig. 28(*d*)) produce large amplitudes of the low-sequency coefficients, and the high frequencies produce large amplitudes of the high-sequency coefficients. Therefore, it is possible to identify the sequency coefficients into which most of the energy associated with frequencies in the video signal spectrum (Fig. 28(*a*)) are transformed. Frequencies from 0 to $\frac{1}{2}f_{sc}$ are converted to the coefficients of sequency 0 to $n - 1$, frequencies from $\frac{1}{2}f_{sc}$ to f_{sc} are converted to sequencies from $n - 1$ to 0, and frequencies from f_{sc} to $(3/2)f_{sc}$ are converted to sequencies 0 to $n - 1$. Therefore, both low frequencies and frequencies near to subcarrier frequency are carried by the low-sequency coefficients of the transform.

The method is very effective with colour signals. For example, the sequency spectrum of 100% colour bars transformed after re-grouping, shown in Fig. 29, has very little energy in the high-sequency coefficients compared with the spectrum for colour bars without re-grouping shown in Fig. 19. However, the spectrum of the alphanumeric characters signal is most adversely affected by re-grouping (compare Fig. 30 with Fig. 18). This is because components having relatively low frequencies, but high amplitudes, are converted to high-sequency coefficients. The bit-rates required for these signals are shown in Fig. 31 for several values of n. Comparing the results for the combined spectrum of colour bars and alphanumeric characters in Fig. 31 with those in Fig. 22 shows that the numbers of bits needed for these signals are consistently higher when

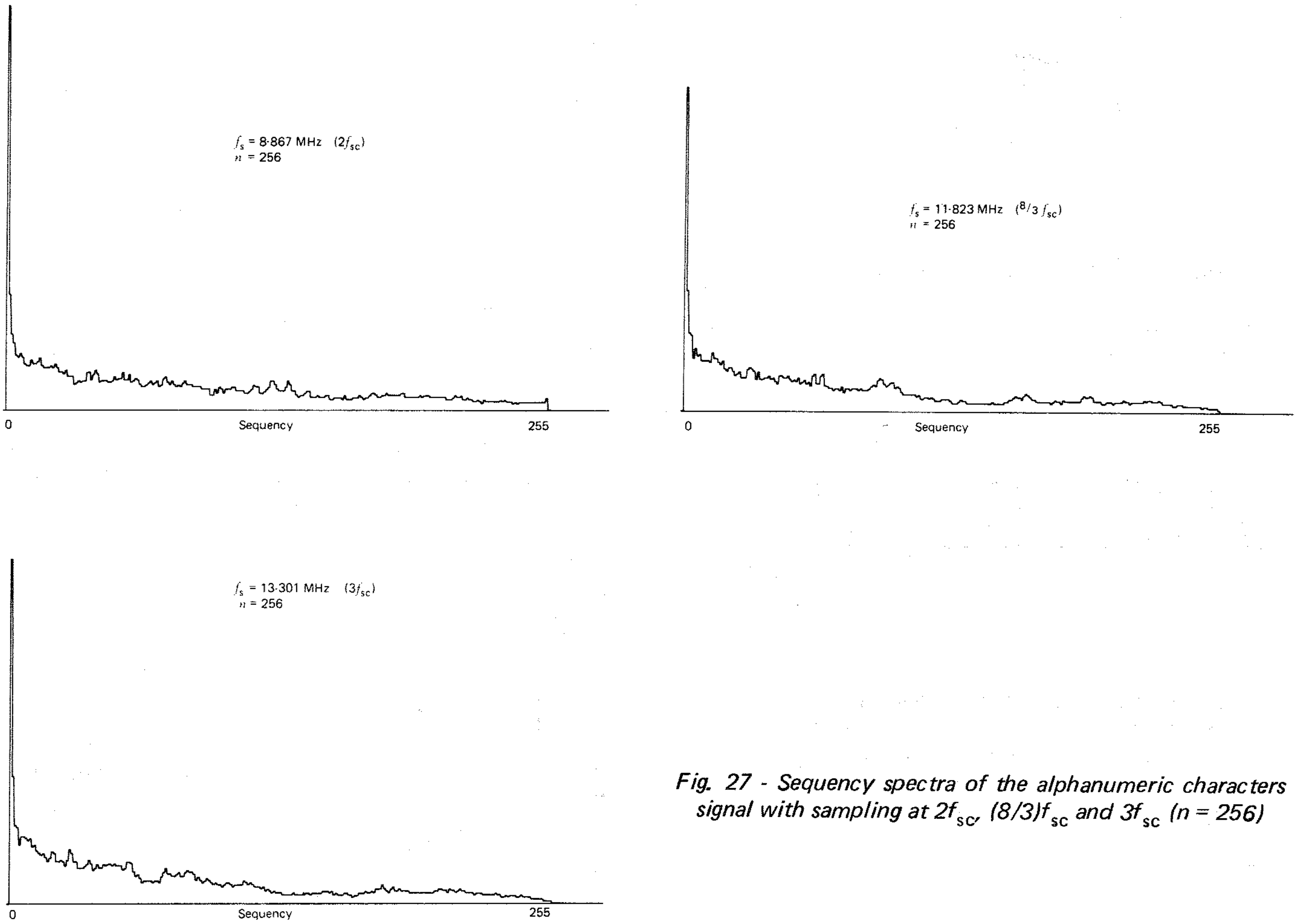

Fig. 27 - Sequency spectra of the alphanumeric characters signal with sampling at $2f_{sc}$, $(8/3)f_{sc}$ and $3f_{sc}$ ($n = 256$)

sub-sampling is used. (This situation is, however, reversed when non-linear quantisation is employed in the Hadamard domain.[13])

5.6. Two-dimensional transformations

Walsh-Hadamard transformation is more effective when samples close together in the signal are similar in value, because the amplitudes of high-sequency coefficients are then small. Since video signals are produced by scanning a two-dimensional picture, samples obtained from neighbouring picture points are generally similar in value. So there is a similarity between samples in the video signal one line period apart, as well as between adjacent samples on the same line.

This similarity can be exploited by grouping together, for transformation, sets of samples which correspond to rectangular areas of the picture. If the number of signal samples in the height and width of each area are both integer powers of two, then the spaces between the scanning lines correspond to transitions in the Walsh functions. It is possible then to describe the Walsh functions in terms of horizontal and vertical sequencies which, to a first approximation, correspond to horizontal and vertical spatial frequencies in the picture. The set of Walsh functions of order 16 are shown in Fig. 32 as two-dimensional (4 x 4) functions identified by components of horizontal and vertical sequency. So, for example, when a two-dimensional transformation is used with signals which do not change from line to line, all coefficients of non-zero vertical sequency have zero amplitude. In general, the majority of the energy in a television signal is carried by coefficients having low horizontal or vertical sequency.

A two-dimensional transformation may be performed by first transforming sets of samples in each line and then transforming the sequency coefficients thus obtained to produce the two-dimensional coefficients. Alternatively, it is possible to derive the sets of two-dimensional coefficients using an ordinary one-dimensional transformation by re-grouping the samples from each area of the picture before transformation. In practice, the latter method is preferable because most of the storage required is for 8-bit television samples, whereas in the former method most of the storage is for transform coefficients which have a greater dynamic range.

The order of the re-grouped samples is that which would be obtained by a sequential scan of the original

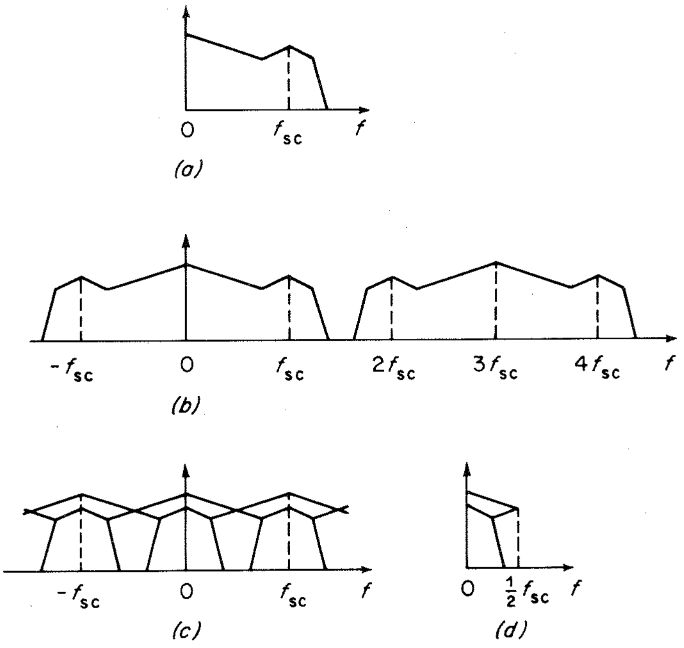

Fig. 28 - A comparison of the frequency spectra produced by sampling at three-times subcarrier rate ($f_s = 3f_{sc}$) with sampling at subcarrier rate ($f_s = f_{sc}$)

(*a*) The baseband spectrum (*b*) $f_s = 3f_{sc}$
(*c*) $f_s = f_{sc}$ (*d*) The effective baseband spectrum for $f_s = f_{sc}$

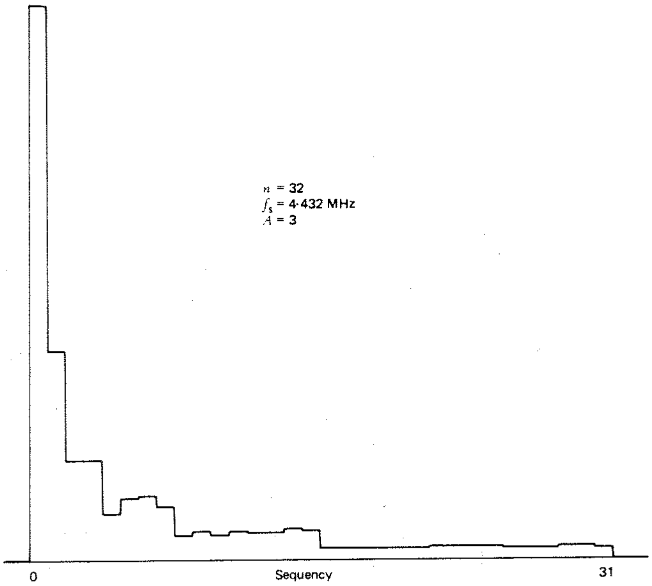

Fig. 29 - The sequency spectrum of a 100% colour bar waveform transformed after sub-sampling at subcarrier rate

area of picture; that is, samples from the first line are taken in order, followed by samples from the second line taken in order, and so on.

When the re-grouped samples have been transformed they are nominally described in terms of the coefficients of a one-dimensional transformation. Therefore, it is necessary to know the relationship between the sequency of the one-dimensional Walsh functions and the horizontal and vertical sequencies used to describe two-dimensional functions. This can be found by investigating the scanned versions of the two-dimensional Walsh functions. For a block of 2^p samples, composed of 2^u samples horizontally and 2^v samples vertically so that $p = u + v$, the one-dimensional sequency can be expressed as a binary number, the digits of which are $s_{p-1}, s_{p-2}, \ldots, s_1, s_0$. Then the horizontal sequency of the corresponding two-dimensional function is the binary number made up of the digits $s_{p-1}, s_{p-2}, \ldots, s_{p-v}$. Further, the vertical sequency of the two-dimensional function is the binary number made up of the digits $s_{u-1}, s_{u-2}, \ldots, s_1, s_0$ if $s_{p-v} = 0$ and $\overline{s}_{u-1}, \overline{s}_{u-2}, \ldots, \overline{s}_1, \overline{s}_0$ if $s_{p-v} = 1$ ($\overline{s}$ represents the complement of s). Applying this to Fig. 32, it is found that going through the

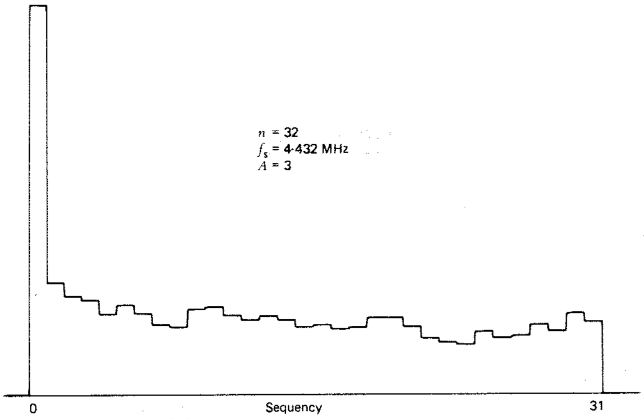

Fig. 30 - The sequency spectrum of the alphanumeric characters signal transformed after sub-sampling at subcarrier rate

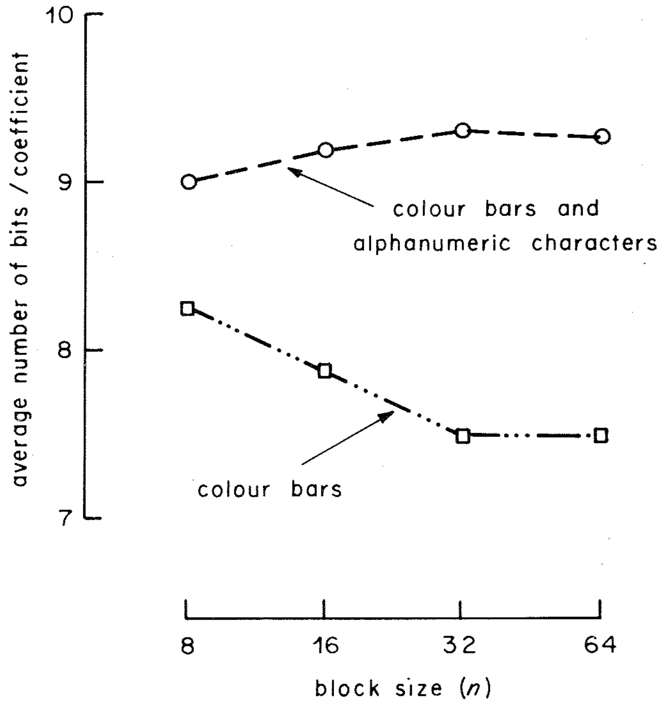

Fig. 31 - Transformation of groups of sub-sampled signals: the variation of bit-rate with transformation order

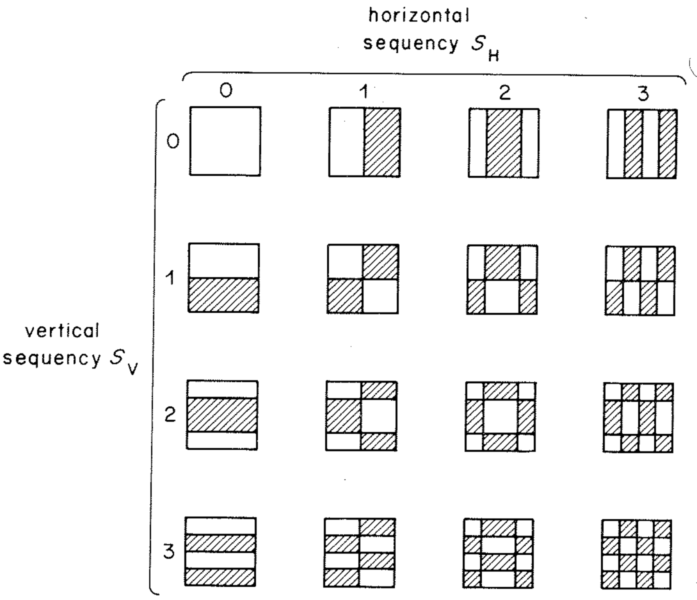

Fig. 32 - The set of 16 basis functions for Walsh-Hadamard transformation of a two-dimensional (4 x 4) block of picture samples. White represents a function value of +1 and the shading represents a function value of −1

functions with $S_H = 0$ with increasing S_V followed by the functions with $S_H = 1$ with decreasing S_V followed by the functions with $S_H = 2$ with increasing S_V followed by the functions with $S_H = 3$ with decreasing S_V is equivalent to going through the one-dimensional sequencies from 0 to 15. This allows the one-dimensional sequency spectra of re-grouped samples to be interpreted in terms of two-dimensional sequencies.

The one-dimensional spectra of the alphanumeric characters signal are shown in Fig. 33 for two-dimensional blocks of 32 samples. In general, the coefficients become smaller as the horizontal and vertical sequencies increase. This is demonstrated more clearly by Fig. 34 which shows the relative numbers of bits required for each coefficient arranged in terms of horizontal and vertical sequency. The models show that the amplitudes of the coefficients decrease more rapidly with increases in horizontal sequency than with increases in vertical sequency. This is because on the picture the distance between adjacent lines in the same field is almost twice the distance between adjacent samples on the same line. So, as the vertical samples are more widely separated, they are less likely to be similar in value.

The one-dimensional spectra of colour bars are shown

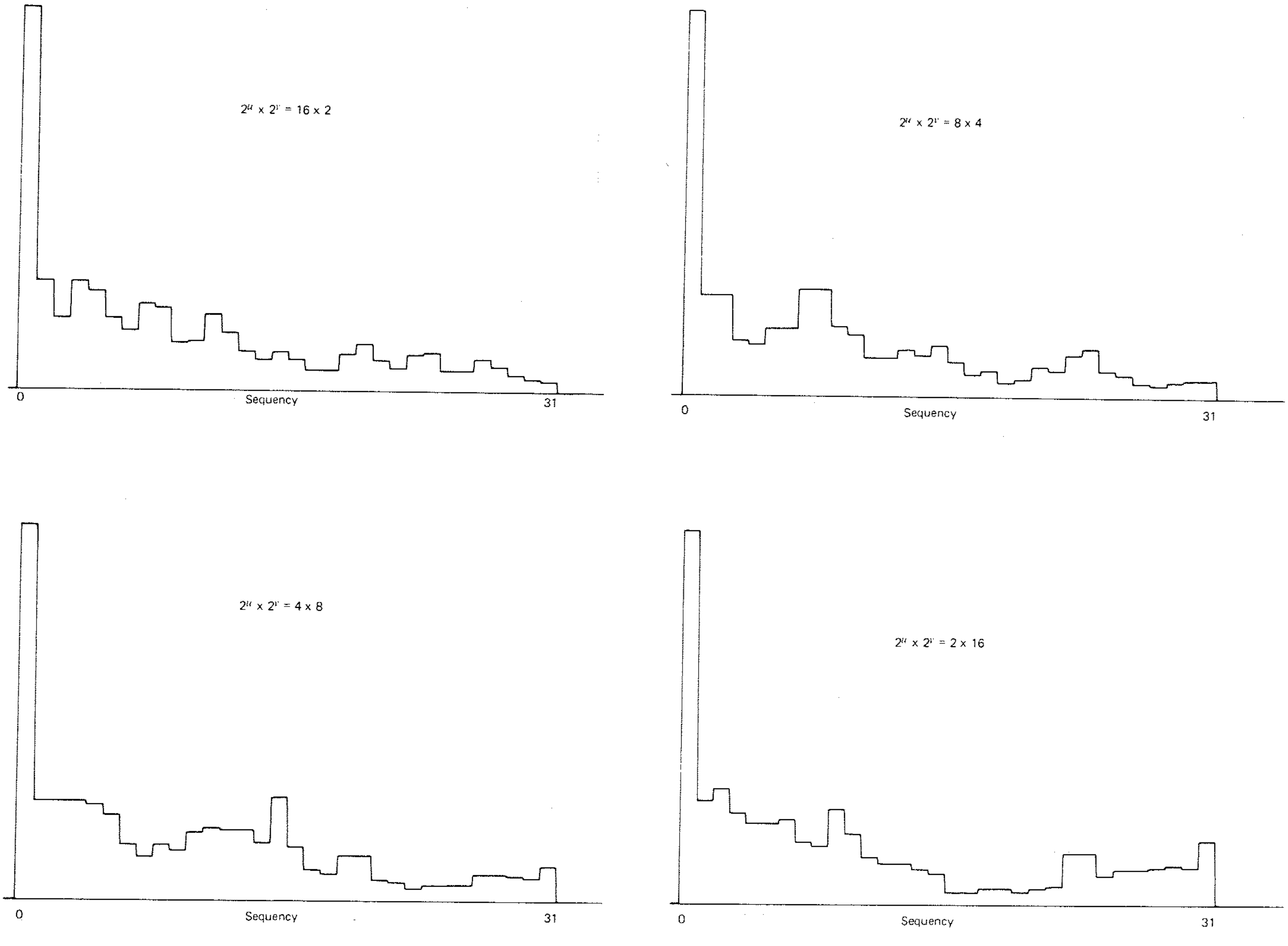

Fig. 33 - Sequency spectra of two-dimensional transforms of the alphanumeric characters signal (f_s = 13·296875 MHz, n = 32)

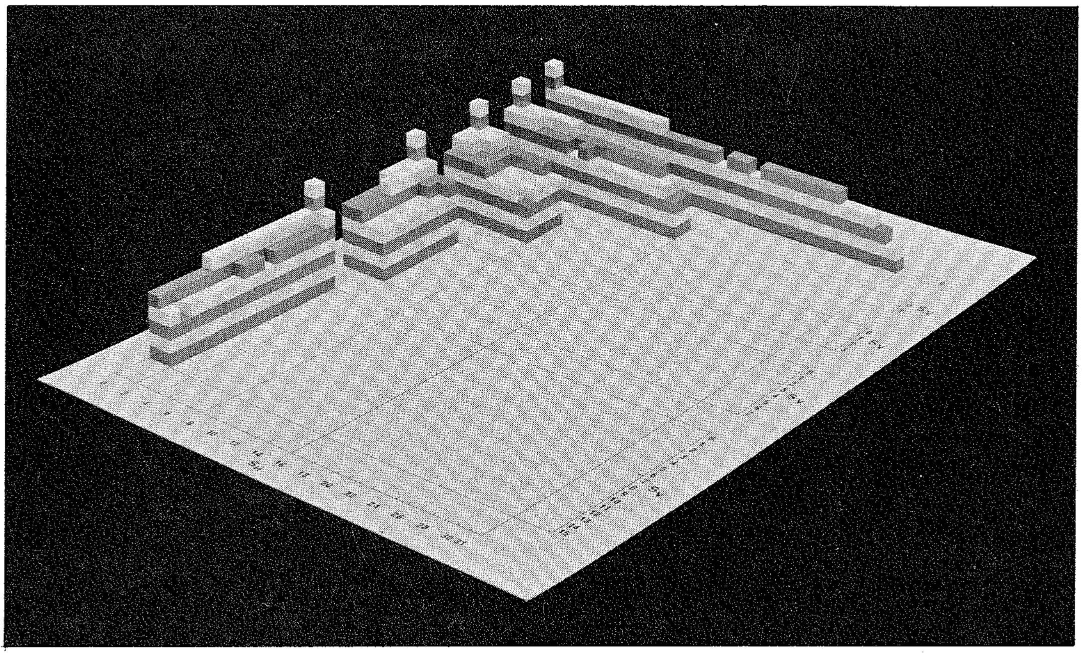

Fig. 34

Sequency spectra for the alphanumeric characters signal showing the profiles of the most significant bits required for the coefficients in terms of horizontal and vertical sequency (f_s = 13·296875 MHz, n = 32). The diagrams demonstrate only the number of most significant bits that may be omitted from the coefficients and should not be interpreted as indicating the number of bits required for each coefficient

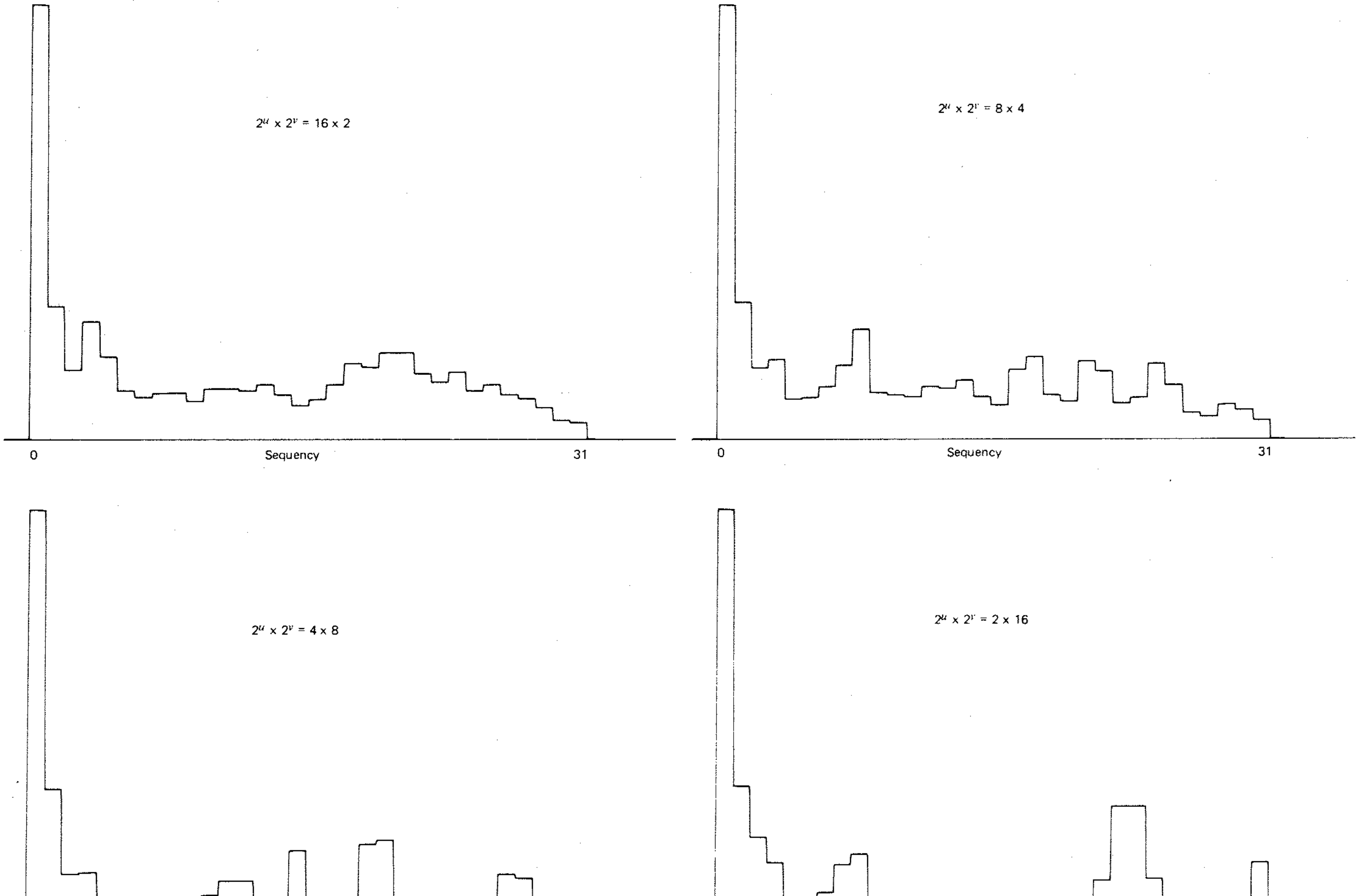

Fig. 35 - Sequency spectra of two-dimensional transforms of a 100% colour bar waveform (f_s = 13·296875 MHz, n = 32)

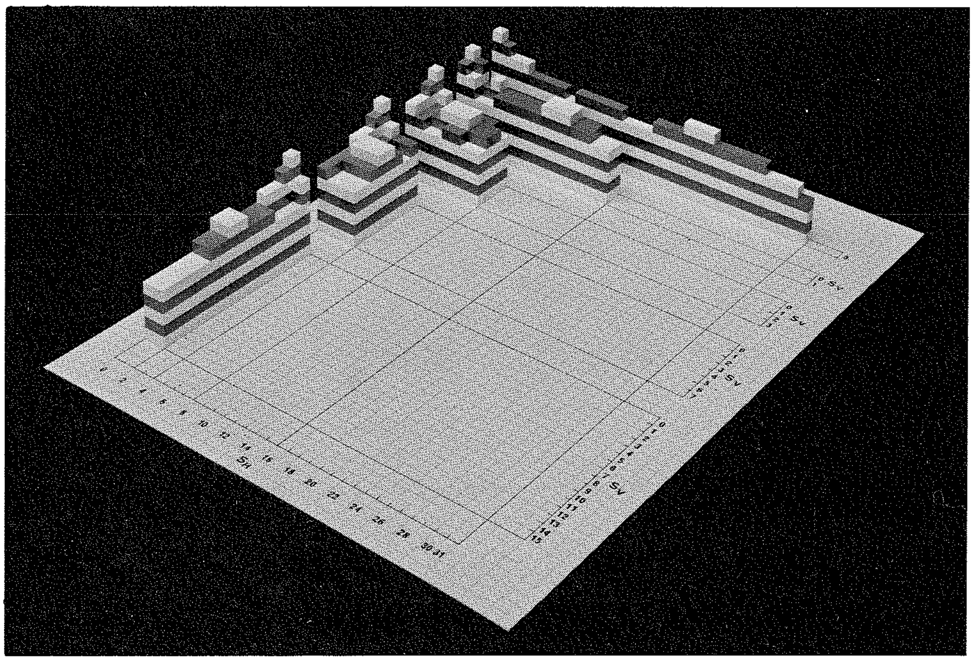

Fig. 36 - Sequency spectra for a 100% colour bar waveform showing the profiles of the most significant bits required for the coefficients arranged in terms of horizontal and vertical sequency (f_s = 13·296875 MHz, n = 32). The diagrams demonstrate only the number of most significant bits that may be omitted from the coefficients and should not be interpreted as indicating the number of bits required for each coefficient

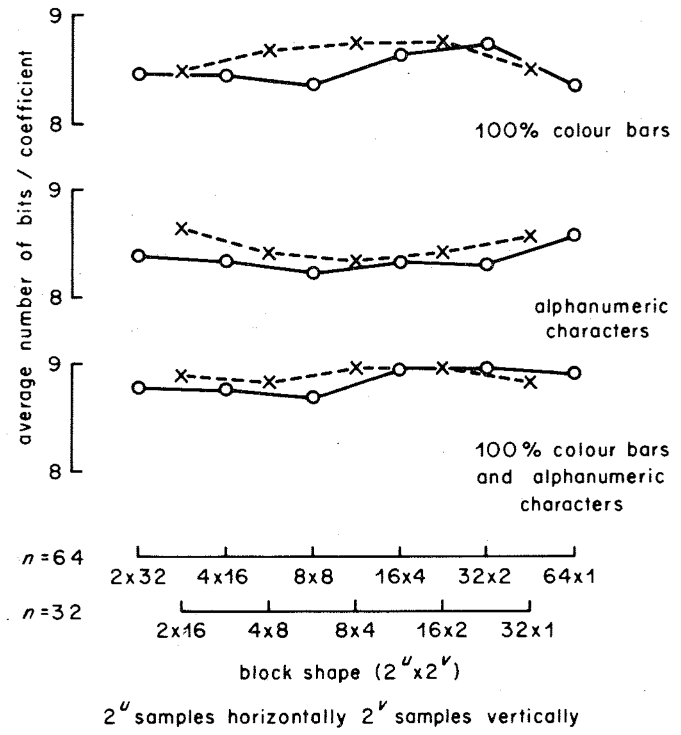

Fig. 37 - Graphs of the average number of bits/coefficient against block shape for two-dimensional transformations. Dashed lines are for n = 32 and continuous lines are for n = 64

in Fig. 35 for two-dimensional blocks of 32 samples. These spectra show large coefficients at sequencies corresponding to one-quarter of sampling frequency in the vertical direction. This is caused by the offset of approximately one-quarter of a cycle between subcarrier waveforms on adjacent lines of the same field. The peak produced by the subcarrier is visible in the models of Fig. 36.

The effect on the bit-rate of changes in the shape of the area of picture transformed is dependent on the signal material. The graphs of bit-rate against block shape for n = 32 and 64 (Fig. 37) show that while square blocks slightly reduce the bit-rate for alphanumeric characters, square blocks may slightly increase the bit-rate for colour bars. When the spectra for the two signals together are considered, there is little advantage in any particular shape.

For larger values of n, the spectra have similar characteristics but more detail can be seen. For example, Figs. 38 and 39 show spectra for blocks of 256 samples. The spectra for blocks of 16 x 16 samples are shown arranged in terms of horizontal and vertical sequency in Figs. 40 and 41. In Fig. 40 the peak at S_V = 7 and 8 and S_H = 9, 10, 11, 12 and 13 was produced by the colour synchronising burst on the back-porch of the television waveform.

5.7. Pseudo-random Hadamard transformation

For some values of n several different Hadamard transformations are possible. For bit-rate reduction, the Walsh-Hadamard transformation is of most interest because of the correspondence between sequency and frequency. However, another Hadamard transformation with contrasting special properties uses the pseudo-random derived matrices mentioned in Section 2.1.

The basis functions for this transformation are formed using cyclically-shifted versions of a pseudo-random sequence. Therefore, the functions have no distinguishing feature such as sequency which may be used to classify them. Also since any phase of a signal is equally possible in the television waveform, it is reasonable to expect this transformation to spread the signal energy evenly between all the functions except for that which describes the average signal level.

One-dimensional pseudo-random transform spectra with n = 32 for alphanumeric characters and colour bar signals are shown in Fig. 42. Apart from the coefficient

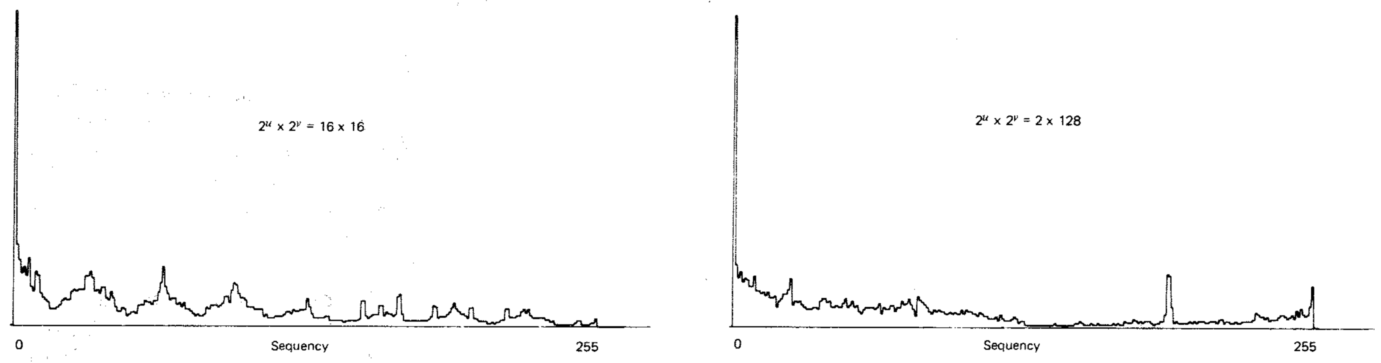

Fig. 38 - Sequency spectra of two-dimensional transforms of the alphanumeric characters signal (f_s = 13·296875 MHz, n = 256)

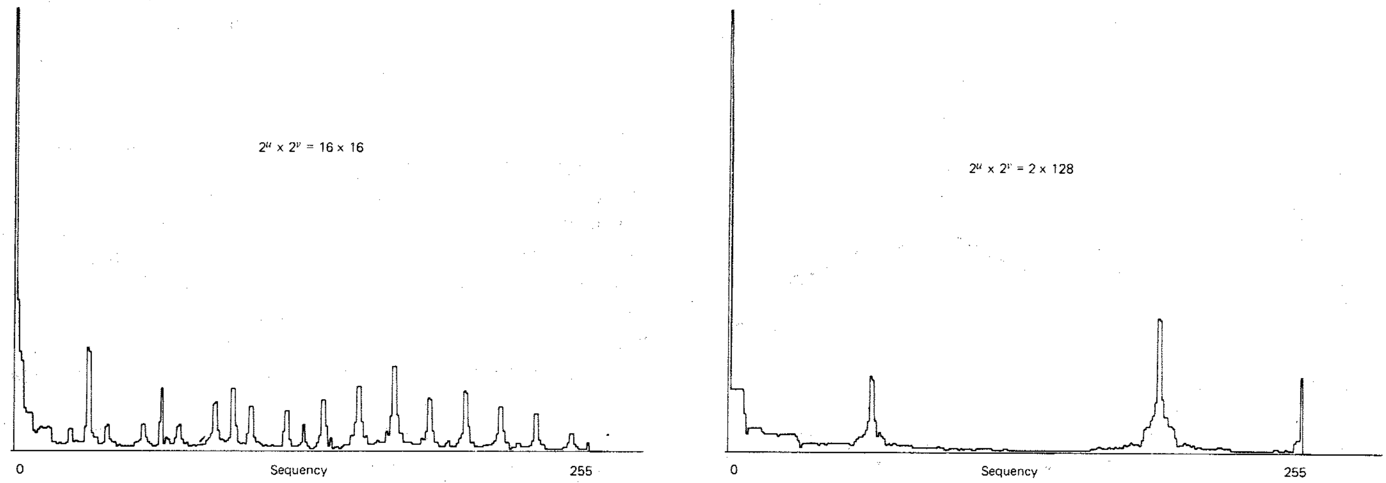

Fig. 39 - Sequency spectra of two-dimensional transforms of a 100% colour bar waveform (f_s = 13·296875 MHz, n = 256)

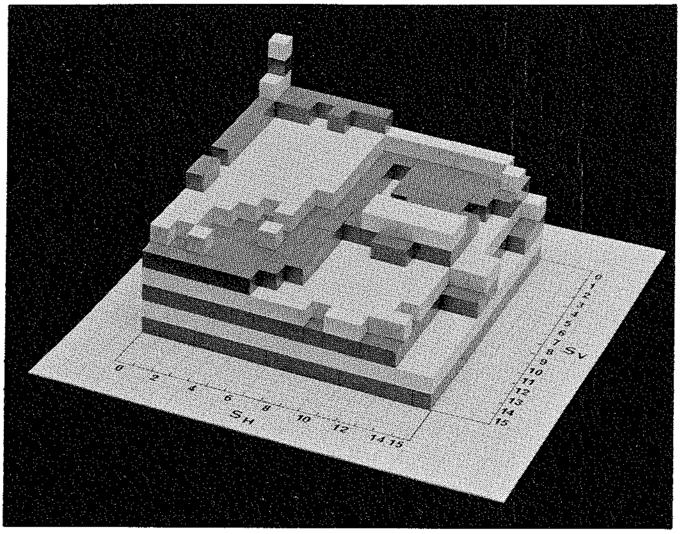

Fig. 40 - The sequency spectrum of the alphanumeric characters signal showing profiles of the most significant bits required for the coefficients arranged in terms of horizontal and vertical sequency (f_s = 13·296875 MHz, n = 256). The diagrams demonstrate only the number of most significant bits that may be omitted from the coefficients and should not be interpreted as indicating the number of bits required for each coefficient

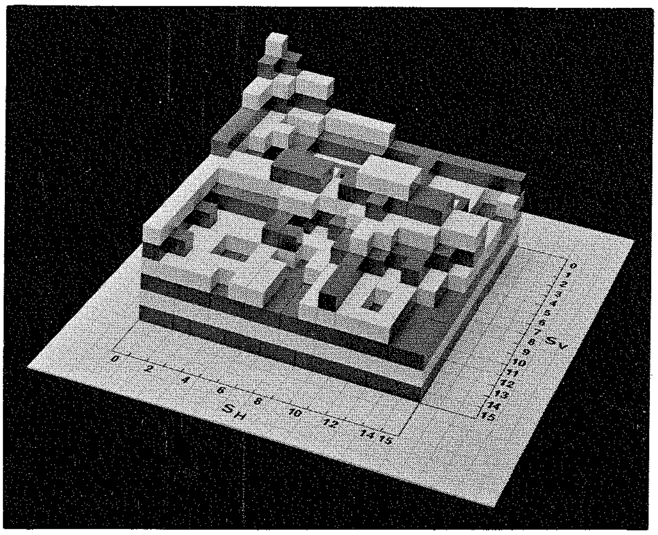

Fig. 41 - The sequency spectrum of a 100% colour bar waveform showing profiles of the most significant bits required for the coefficients arranged in terms of horizontal and vertical sequency (f_s = 13·296875 MHz, n = 256). The diagrams demonstrate only the number of most significant bits that may be omitted from the coefficients and should not be interpreted as indicating the number of bits required for each coefficient

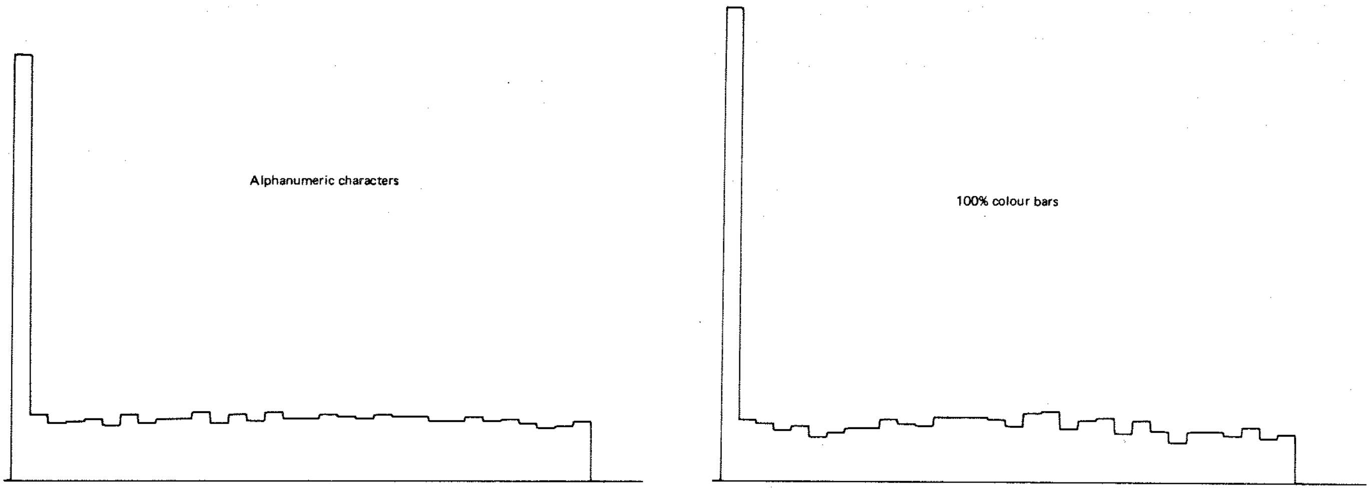

Fig. 42 - Spectra for the pseudo-random Hadamard transformation (f_s = 13·296875 MHz, n = 32)

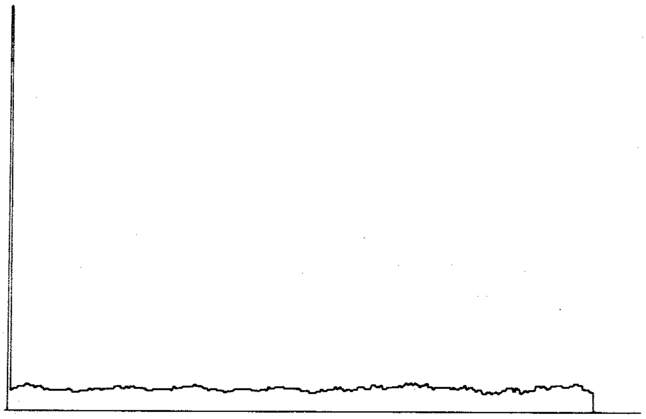

Fig. 43 - Spectrum of a 100% colour bar waveform for the pseudo-random Hadamard transformation (f_s = 13·296875 MHz, n = 256)

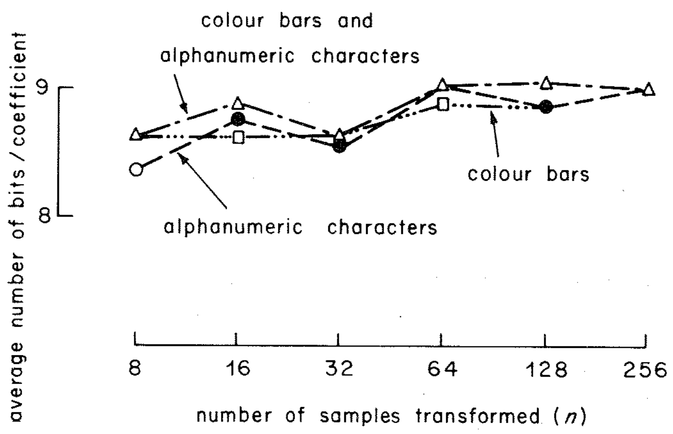

Fig. 44 - The variation of the average number of bits/ coefficient with the transformation order for the pseudo-random Hadamard transformation

describing the average level, all the maximum amplitudes are within a 2:1 range of each other. The spectra for the two different signals are much more similar than the equivalent spectra for Walsh-Hadamard transformation. Increasing the order of the transformation makes the coefficients smaller relative to that describing the average level of the signal. Fig. 43 shows the pseudo-random Hadamard spectrum of order 256 for a 100% colour bar signal.

The average number of bits required to represent each coefficient in the pseudo-random Hadamard transform is shown as a function of n in Fig. 44. The number of bits required are generally higher than in the equivalent Walsh-Hadamard transformation (Fig. 22). The low results for n = 32 have occurred because, with this size of transformation, many of the coefficients reached almost to the top of the range of a bit. This effect is less noticeable in the results for Walsh-Hadamard transformation because there is a much wider spread of coefficient values.

6. Discussion of results

In the preceding sections, it was first shown that considerably more bits are required to represent the transform coefficients than are required to represent the original samples. Then methods of omitting some of the least significant bits were presented which remove half of this increase. In Section 5, it was shown that a further reduction could be made from the most significant bits using several different transformation arrangements. However, in spite of these reductions, if a wide range of signals is to be reproduced accurately, then the bit rate required for the transform is still greater than the original bit rate. Therefore, the transformation itself does not lead directly to a reduction of the bit rate, but must be combined with non-linear or adaptive coding methods if any reduction is to be achieved.[13]

It is appropriate here to examine in more detail the purpose of transformation in bit-rate reduction. The redundancy in the signal arises because the brightness of a point in the picture is usually closely related to the brightness of the surrounding areas. To expose this redundancy, the relationships between the values of nearby samples should be described, rather than the sample values themselves. This can be done by transforming the signal in blocks in such a way that the signal is represented by the amplitudes of a set of basis functions. The use of orthogonal basis functions ensures that no part of the signal energy is carried twice by the transform.

The effectiveness of a transformation can be judged by the way in which the signal energy is split between the transform coefficients. A transformation performs well if most of the energy from commonly occurring signals is concentrated into a small number of coefficients. These important coefficients can then be coded accurately. The remaining coefficients will have small values for commonly occurring signals, but, to deal with more unusual signals, the capacity must be provided for occasional, large values. This can be done using relatively few bits by non-linearly quantising the coefficients.

Walsh-Hadamard transformation concentrates most of the energy of commonly occurring luminance signals into the low-sequency coefficients. The zero-sequency coefficient is always very large, whereas the other low-sequency coefficients can vary between large and small values. The distribution of these values is strongly dependent on the number and the amplitude of edges in the picture. Luminance signals rarely produce large high-sequency coefficients.

However, with sampling frequencies near to three-times the colour subcarrier frequency, the energy of colour signals is spread throughout the medium- and high-sequency coefficients. For signals representing areas of constant colour, the amplitudes of the coefficients are approximately uniformly distributed between zero and the maximum value, with a small bias towards the large values.

So, whereas the Walsh-Hadamard transform of monochrome television signals is conditioned reasonably well for bit-rate reduction, the transform of PAL-encoded colour signals, sampled at three-times subcarrier frequency, is not so suitable.

Although none of the transformation systems described in Section 5 leads directly to a reduction in bit-rate, some of the systems do improve the distribution of the signal energy between the coefficients. Thus they may allow worthwhile reductions to be made by non-linear quantisation or adaptive coding.[13]

The greatest improvement can be made by using a sampling frequency of $2f_{sc}$* or $(8/3)f_{sc}$, since this concentrates the colour signal energy into a small number of coefficients as well as reducing the word rate. However, reducing the sampling frequency will cause the high-sequency coefficients produced by luminance signals to be slightly increased.

Transformation of two-dimensional blocks concentrates luminance signals into the low horizontal and vertical sequencies and colour signals into the mid-vertical sequencies.

There is less benefit in the other arrangements tried in Section 5. It is reasonable to expect that increasing the transformation order would reduce the occurrence of large values in most coefficients, but, beyond one-dimensional transformations of 32 or 64 samples, such reductions would be unlikely to be sufficient to justify the increases in cost. Further, the sub-sampling arrangement described in Section 5.5 is beneficial for colour signals, but detrimental for luminance signals because of the much wider spacing of the samples to be transformed. On balance, this method would produce an improvement because large amplitude coefficients occur less frequently, and there is therefore better scope for non-linear coding.

In the pseudo-random Hadamard transformation, the signal energy is widely spread between the coefficients and the distribution of amplitudes within each coefficient is not favourable for non-linear quantisation. Moreover, no fast algorithm is known for this transformation.

This suggests that a system which combines $2f_{sc}$ sampling with two-dimensional Walsh-Hadamard transformation would have most potential for bit-rate reduction. The sub-Nyquist sampling method does, however, bring some penalties in picture quality,[10] which might rule out its use in high-quality transmission. In this case, the use of $(8/3)f_{sc}$ sampling, which is less effective at concentrating the colour but could work satisfactorily with sharp cut-off filters,[11] would remain as the best alternative. Otherwise, for three-times subcarrier sampling, the sub-sampling method works well except in areas containing large-amplitude luminance transitions. This method could be combined with two-dimensional transformation, in which case the spectrum would take on the characteristics of both the spectra for the sub-sampled transformation and the two-dimensional transformation. The colour signal energy would be concentrated into the coefficients with low horizontal and middle vertical sequencies, whilst the luminance signal energy would appear mostly in the coefficients with low horizontal or vertical sequency. Luminance signals would produce larger amplitudes in the higher horizontal sequencies than with the two-dimensional transformation on its own, because of the increased horizontal separation of the samples in sub-sampled signals.

Sampling at four-times subcarrier frequency would concentrate most of the colour signal energy into the quarter-sampling frequency coefficients although it would increase the word rate. If $4f_{sc}$ were used, it is possible that some of the highest sequency coefficients could be omitted altogether to offset the increased word rate.

All the methods which have been described of concentrating the colour signal energy have some adverse effect, either on the distribution of luminance signal energy in the transform or on the word rate of the signal. If Fourier transformation were to be used, then the distribution of both colour and luminance signal energy would be improved, although more complicated transformation equipment would be required.

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* This is possible if a method of sub-Nyquist sampling[10] is used.

It has been possible to check some of the results and assertions presented here using a real-time Walsh-Hadamard transformer.[12] Groups of 32 samples taken sequentially and groups made up by sub-sampling at colour subcarrier rate were transformed. Also, a range of sampling frequencies from twice to three-times subcarrier frequency has been used. In all cases, the number of most significant bits which were found to be unused was exactly as measured with the Walsh spectrum analyser.

It was found, using the real-time transformer, that errors caused by the value of a coefficient exceeding the range allocated to it can be tolerated provided that the impairment is small and infrequent. For groups of sequential samples taken at three-times subcarrier rate, very few bits are used sufficiently infrequently to be removed without causing impairment, but, when sub-sampling is used, a larger number of bits may be removed. This is because high values occur much less frequently in the sub-sampled transform. Further, for this reason, non-linear quantisation cannot be used on the sequentially-grouped transform of colour signals with sampling at three-times subcarrier rate, but can be used successfully with the sub-sampling transformation.

The results obtained with the real-time transformation system will be presented more fully in another report.[13]

7. Conclusions

Transforms of picture signals describe the relationships between picture elements instead of their individual values. The values of nearby picture elements are usually related, so it may be easier to reduce the bit-rate of the transform than to reduce that of the original signal.

Walsh-Hadamard transformation is particularly attractive for this purpose because the operations involved are simpler than for other transformations. The transformation process has been described mathematically and illustrated with a typical example. The most efficient fast-transformation algorithm for television signals has been explained and a method of implementing the algorithm in digital circuits using a minimum of storage has been described.

Although transformation increases the number of bits needed to represent a signal to full accuracy, the most significant bits of some coefficients are seldom or never used. The numbers of bits which are not used have been measured using a Walsh spectrum analyser for a wide range of signals. Signals containing large numbers of fast transitions between black-level and white-level and signals containing high amplitudes of colour subcarrier produce the largest coefficients and therefore require more bits than other signals. Some of the least significant bits may be omitted because the amounts of energy which they represent are negligibly small. However, the number of most and least significant bits which can be omitted is not by itself sufficient to offset the increase caused by the transformation. With this form of coding, it is expected that an average of between 8·5 and 9·0 bits per coefficient would be required to maintain the same accuracy as 8-bit p.c.m.

Ideally, a transformation should concentrate most of the energy of commonly occurring signals into a small number of transform coefficients. Then, the remaining coefficients will usually take small values, although occasionally they may take large values for unusual signals. A transform with these characteristics can be coded efficiently by describing the important coefficients accurately and then using non-linear quantisation or an adaptive system to describe the other coefficients.

For luminance signals, Walsh-Hadamard transformation of sequentially grouped samples produces a distribution of energy in the transform which is reasonably suitable for bit-rate reduction. Most of the signal energy is carried by the low-sequency coefficients and the other coefficients take large values only rarely. However, the energy from colour subcarrier signals with sampling at three-times subcarrier rate is spread throughout the medium- and high-sequency functions. Large values of these coefficients are produced for a high proportion of the time, so that the transform is unsuitable for bit-rate reduction.

Increasing the order of the transformation greatly increases the cost of the transformation equipment. It appears that for one-dimensional transforms of blocks of more than 64 samples the savings of bit-rate to be made are insufficient to justify the increase in cost.

The distribution of energy in the transform can be changed by altering the rate at which the analogue signals are sampled. Choosing sampling frequencies such as twice or eight-thirds-times subcarrier frequency concentrates most of the colour signal energy into a small number of coefficients. Then the bit-rate of the other coefficients can be reduced more easily.

Another method of improving the energy distribution is to transform groups of samples corresponding to two-dimensional blocks of picture elements. Then most of the luminance signal energy is concentrated into the low horizontal or vertical sequencies and the colour signal energy is confined to the middle vertical sequencies. An improvement is achieved because television signals are usually strongly related from line to line. This method could be combined with twice or eight-thirds-times subcarrier-frequency sampling to further improve performance for colour signals.

The maximum concentration of energy in the transform results if the basis functions are matched to the most commonly occurring television signals. Although Walsh functions are moderately good in this respect, it is expected that sinusoidal functions would be better. Therefore, Fourier transformation could lead to improved performance, but at the cost of more complicated transformation equipment.

Where possible, the assertions presented have been checked using a real-time transformation system. The performance of the real-time system is in close agreement with the performance predicted from measurements obtained by Walsh spectral analysis.

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9. Appendix

A Method of Reducing Redundancy in the Walsh-Hadamard Transform of Binary-Coded Signals

Because of the nature of the Walsh-Hadamard transformation, the sets of code-words used to describe the transform of the signals contain redundancy. This Appendix presents a method by which the redundancy may be reduced without losing the characteristics of the transformation.

At each stage in the fast transformation, two numbers, say A and B, are added and subtracted. Now if A and B are both even, or if A and B are both odd, both the sum and the difference will be even. If A is odd and B is even, or if A is even and B is odd, then the sum and the difference will be odd. In terms of binary numbers this means that the least significant bits of the sum and the difference are always related, and with some representations of negative numbers are equal. Clearly, it is inefficient to transmit both least significant bits, so only one, say that of the sum, need be sent to the next stage of the transformation. At the corresponding stage in the inverse transformation, where $A + B$ and $A - B$ are added and subtracted to form $2A$ and $2B$, the appropriate value of the least significant bit of the difference can be formed from that of the sum, to ensure that the inverse transformation reproduces the input values exactly.

If this operation is performed at each stage in the transformation, then the number of bits omitted from each coefficient is equal to the number of difference operations used to calculate the coefficient. Also, errors are introduced only at the difference operations, so the maximum value of the error in a coefficients falls within the range of the omitted bits. Therefore, the transform could be calculated in the normal way and truncation carried out in the transform domain. Then it would be possible to return to the exact signal values using a modified inverse transformation in which the correct values of the least significant bits are re-inserted at each stage.

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